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# Reduced Order and Observer-Based Reset Control Systems with Time Delays

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ABSTRACT

This paper establishes a new mechanism to stabilize plants using reduced order reset controllers. The proposed method uses state feedback to change the dynamics of plants to guarantee oscillation behavior instead of stability, then the reset mechanism will lead to stability. We show that the base system could be unstable while the reset mechanism drives the states to the equilibrium point. The order of the reset controller equals the rank of the plant's input matrix. We show that the controller dynamics force some states to converge to the equilibrium point within a finite time. The behavior of the rest of the plant's states depends greatly on the selection of the state feedback gain which can be selected by any appropriate conventional method. Moreover, the stability of reset time-delay systems is addressed based on a similar theorem of the Lyapunov-Krasovskii theory. Sufficient conditions are given in terms of linear matrix inequalities to guarantee asymptotic stability of the overall dynamics. Simulation results are presented to demonstrate the effectiveness of the proposed reset approaches.



## 1. Introduction

Dynamical systems usually demand-control theory to carry out user specifications in terms of performance and stability. Most of the control methodologies that have been investigated over decades are devoted to analyzing control laws that have similar nature to the controlled process. On one hand, technological advancement allows researchers to utilize controllers with continuous and discrete dynamics, i.e. mixed dynamics. Such systems exhibit continuous dynamics and discrete jumps. This class of controllers, reset controllers, provides advantages over the conventional ones since it has richer dynamics. On the other hand, the use of controllers that compromise dynamics different from the controlled plant leads to complexity, and its implementation might become more sophisticated.

Recently reset or impulsive control has been addressed through various works in an attempt to provide flexible tools to a good performance [1], [2], [3], [4], [5], [6], [7]. The origin of reset control systems dates back to the nonlinear Clegg integrator, see [8]. The Clegg controller was investigated in order to overcome the time lag appeared due to linear integrators. The reset condition accompanied with Clegg integrator that causes the state to vanish usually is satisfied when its input vanishes. Another important advancement of reset systems, first order reset element (FORE), is introduced by Horowitz [9]. FORE is simply modeled by a first order continuous dynamics with a particular reset mechanism. Stability of reset time delay systems can be addressed based on a similar theorem of Lyapunov-Krasovskii theory. The idea of reset techniques have attracted attention of many researchers [10], [11], [12], [13], [14], [15].

The reset surface can be defined as a monotone sequence of time, and the dynamics can be treated as an impulsive dynamical system [16]. Generally, reset control framework is based on the theory of impulsive control systems. Reset control systems can be characterized by three components, continuous flow dynamics, discrete jumps, and reset rule. The continuous flow is defined by a set of differential equations that characterize the evolution of the system between consecutive jumps. The discrete jumps represent the abrupt change in the trajectories. Reset rule governs the switching between the discrete jumps and continuous flow. Although the design of reset controllers could improve the performance and stabilize unstable plants, it has been clear that the design procedures must be done with care because it may degrade the performance and destabilize processes [17]. As reported in literature, reset control systems represents a jump behaviour in which the system state resets at the instant it crosses a defined reset surface. Reset control systems are similar in some sense to switched systems, are classified as a particular class of hybrid dynamical systems [18]. Although both types of control systems are hybrid but are intrinsically different. With this in mind, reset control systems represents a discontinuous behaviour in which the system states reset when the reset surface is hit.

Lyapunov based approach [19], [20], [21], [22], and also passivity based methods [23], [24], [25] are used to investigate the stability of time delay rest systems. Some theoretical tools is devoted to investigate reset systems with time-delay in [26] according to Lyapunov-Krasovskii theory. The design of control laws are usually used to improve the performance, but the reset technique should be designed carefully since it may destabilize the systems. Although reset controllers show more advantages over conventional linear controllers, several drawbacks and challenges could arise. On one hand, reset controller may destabilize a stable base control system if inappropriate reset sequence is applied. This means that reset systems must be designed carefully in order to improve the performance specifications. On the other hand, it might be there exists linear controller performs better than the reset controller. However, linear control theory have more limitations that makes the design of linear controller very challenging or impossible [12].

The methodology of designing a rest control law usually makes more demands than other classical ones. These demands emerge in selecting the reset laws and the states to be reset as well as the dynamics of its continuous flow. This paper proposes relatively simple procedures that can stabilize the system and decreases the settling time efficiently. While concentrating on developing reset systems methods, the main contributions of this article are:

1. The proposed reset technique considers base dynamics which is not necessary stable, different from existing results that assumes stability of the base dynamics.
2. Our methodology forces a number of states to reach the equilibrium point in a finite time whereas most of the existing reset-controllers require infinite time to drive the states to the equilibrium.
3. The proposed method adds a substantial contribution in terms of reset control design rather than on reset analysis and hence our approach can be remarkably used to design more effective reset controllers to meet a predefined performance objective.
4. An extending results are given to investigate observer-based reset-control systems with delay-delays. Sufficient conditions in terms of linear matrix inequalities are derived to obtain controller and observer gain matrices.

Notations: In this paper, $\mathbb{R}^n$ represent the n-dimensional Euclidean space. We use $S^T$ and $S^{-1}$ to represent the transpose and inverse of the matrix, respectively. We use $S > 0$ to represent a symmetric positive definite 2 matrix. The $n$ dimensional identity matrix is denoted by $I_n$. Matrices are assumed to have compatible dimensions if their dimensions are not explicitly stated. In symmetric real block matrices, we use the symbol • to denote elements that are induced by symmetry.

## 2. Problem Formulation

Consider a plant described by the following linear time invariant dynamics:

$$\dot{x}(t) = A_p x(t) + B_p u(t) \tag{1}$$

where $A_p \in \mathbb{R}^{n\times n}$, and $B_p \in \mathbb{R}^{n\times m}$. The input matrix $B_p$ is assumed to be a full column rank matrix. The proposed reset controller that stabilizes the plant (1) is given by the following dynamics.

$$\begin{aligned} \dot{x}_r(t) &= A_{r1}x_p(t) + A_{r2}x_r(t) \\ x_r\left(t^+\right) &= A_{\rho 1}x_p(t) + A_{\rho 2}x_r(t) \end{aligned} \tag{2}$$

where $x_r(t) \in \mathbb{R}^m$ is the state vector of the controller, $A_{r1} \in \mathbb{R}^{m\times n}$, and $A_{r2} \in \mathbb{R}^{m\times m}$. Since $x_r(t)$ is the controller state, it can be used as a fully resettable state. $x_r\left(t^+\right)$ represents the after reset state, which is updated when some predefined condition holds. The dynamics of the reset controller and the plant can be written as:

$$\begin{cases} \dot{x}(t) = A_p x_p(t) + B_p u(t) & \\ \dot{x}r(t) = Ar1x_p(t) + A_{r2}x_r(t), & (x_p, x_r, \tau) \notin \mathcal{R} \\ \dot{\tau}(t) = 1, & \\ & \\ x_p(t^+) = x_p(t), & \\ x_r(t^+) = A_{\rho 1}x_p(t) + A_{\rho 2}x_r(t) & (x_p, x_r, \tau) \in \mathcal{R} \\ \tau(t^+) = 0, & \end{cases}$$

where $\mathcal{R}$ is the reset surface and $\tau(t)$ is a time regulation parameter. The parameter $\tau(t)$ is used to regulate the reset instants in order to avoid Zeno behavior. Zeno solution occurs if the state encounters the reset surface more than once at the same instant. The reset mechanism is activated immediately when the trajectory of the reset system hits the surface $\mathcal{R}$. Let $u(t) = Kx_p(t) + Lx_r(t)$, then the overall reset control system becomes

$$\begin{cases} \dot{x}(t) = Ax(t), \dot{\tau}(t) = 1, & (x_p, x_r, \tau) \notin \mathcal{R} \\ x\left(t^+\right) = A_R x(t), \tau\left(t^+\right) = 0, & (x_p, x_r, \tau) \in \mathcal{R} \end{cases} \tag{3}$$

where

$$x(t) = \begin{bmatrix} x_p(t) \\ x_r(t) \end{bmatrix}, A = \begin{bmatrix} A + B_pK & B_pL \\ A_{r1} & A_{r2} \end{bmatrix},$$

and

$$A_R = \begin{bmatrix} I & 0 \\ A_{\rho 1} & A_{\rho 2} \end{bmatrix}$$

The first equation in (3) represents the base system or the so called continuous flow dynamics while the second equation describe the jump behaviour at reset instants. The asymptotic stability of the closed loop reset system (3) depends greatly on the structure of its base system. It can be easily verified that if $A$ is Hurwitz and $A_R$ is Schur then the closed loop system is asymptotically stable.

Moreover, stability of the overall reset system (3) is equivalent to the stability of its base system if the reset mechanism is not activated at all. However, reset systems with an inappropriate reset sequence may become unstable even if the base dynamics is stable. On the other hand, a reset mechanism should be applied if the base system is unstable. To our best knowledge, most of existing results study reset control methods when the base system is stable. In this paper, new methods are proposed to stabilize reset systems with unstable base dynamics. In addition, a reduced order controller is given based on the aforementioned stabilization criterion.

For the time delay reset control case, consider the general dynamics whose continuous flow comprises time delay:

$$\begin{cases} \dot{x}(t) = f\left(t, x_t\right) \\ \dot{\tau}(t) = 1, \end{cases} \quad (t, x_t, \tau(t)) \notin \mathcal{R}$$
$$\begin{cases} x\left(t^+\right) = I(t) \\ \tau\left(t^+\right) = 0, \end{cases} \quad (t, x_t, \tau(t)) \in \mathcal{R} \tag{4}$$
$$\begin{cases} x(t) = \phi(t), \\ \tau(t) = 0, \end{cases} \quad t \in [t - \tau^\star, t]$$

where $x(t) \in \mathbb{R}^n$ is the state, $x_t = x(t+\theta)$ denotes the state segment, $\theta \in [-\tau^\star, 0]$, $\tau^\star$ represents the maximum time delay, $f(t,x)$ is a general Lipschitz function, $I(t)$ gives the state's value when the reset takes place, $\phi(t)$ represents the initial condition, and the set $\mathcal{R} \subseteq \{\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}\}$. The following proposition with appropriate Lyapunov functional will be used to show the asymptotic behaviour of time delayed reset systems.

*Proposition 1*: [26] Suppose $f : \mathbb{R} \times \mathcal{C}\left([-h, 0], \mathbb{R}^n\right) \to \mathbb{R}^n$ in (4) maps $R \times$(continuous functions) into bounded sets of $\mathbb{R}^n$, and $u, v, w : \mathbb{R}_+ \to \mathbb{R}_+$are continuous non decreasing functions such that $u$ and $v$ are class $\mathcal{K}$ functions. If there exists a continuous Lyapunov Krasovskii functional for all solutions $x_t$ of (4) except at the rest instants. $V : \mathbb{R}_+ \times \mathcal{C}\left([-h, 0], \mathbb{R}^n\right) \to \mathbb{R}_+$such that

$$u(\|\phi(0)\|) \leq V(t, \phi) \leq v\left(\|\phi\|_c\right) \tag{5}$$

for any $\phi$ satisfies $\|\phi\|_c \leq \gamma$ for some positive scalar $\gamma$, and in addition, the time derivative and the difference of $V$ satisfies the following inequalities:

$$\dot{V}\left(t, x_t\right) \leq -w\left(\|x_t(0)\|\right), (x_t, \Delta(t)) \notin \mathcal{R} \tag{6}$$

$$\Delta V\left(t, x_t\right) \leq 0, (x_t, \Delta(t)) \in \mathcal{R} \tag{7}$$

then the trivial solution of the delay reset system of (4) is uniformly stable. If, in addition, $w(s)$ is positive for positive $s > 0$ then the system is uniformly asymptotically stable, where the norm $\|x_t\|_c = \max_{\theta \in [-\tau^\star, 0]} \|x_t(\theta)\|$.

In the sequel, the stability of linear time invariant system with reset mechanism is analysed. Consider the following time delay reset system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ +A_1 x(t-d) + B_1 u(t-h) + Dw(t), \quad x(t) \notin \mathcal{R} \\ x\left(t^+\right) = A_{\rho 1} x(t) \quad x(t) \in \mathcal{R} \\ y(t) = C_1 x(t), \quad z(t) = E_1 x(t) \end{cases} \tag{8}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control law, $w(t) \in \mathbb{R}^q$ is the disturbance which belongs to the space of energy signals, i.e. $w(t) \in \mathcal{L}_2[0, \infty]$, $d$ and $h$ represents the amount of delay

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in the state and at the input of the plant, respectively. $y(t) \in \mathbb{R}^p$ is the output of the plant and $z(t)$ is the controlled output. The proposed reset observer-based controller is given as

$$\begin{cases} \dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}[y(t) - C\hat{x}(t)] & \\ \quad +C_1\hat{x}(t-d) + C_2\hat{x}(t-h), & \hat{x}(t) \notin \mathcal{R} \\ \quad \hat{x}\left(t^+\right) = A_{\rho_2}x(t) + A_{\rho_3}\hat{x}(t) & \hat{x}(t) \in \mathcal{R} \\ \qquad u(t) = K\hat{x}(t) & \end{cases} \tag{9}$$

where $\hat{x}(t)$ is the estimate of the state $x(t)$. Let the estimation error be defined by $e(t) = \hat{x}(t) - x(t)$ and the augmented vector $\xi(t) = \left[x^T(t), \quad e^T(t)\right]^T$. Then the closed loop system corresponding to (8) and (9) is given by the following state space model:

$$\begin{cases} \dot{\xi}(t) = A_c\xi(t) + B_{c1}\xi(t-d) & \\ \quad +B_{c2}\xi(t-h) + D_w w(t), & \xi(t) \notin \mathcal{R} \\ \quad \xi\left(t^+\right) = A_R\xi(t) & \xi(t) \in \mathcal{R} \\ \quad z(t) = E\xi(t), & y = C\xi(t) \end{cases} \tag{10}$$

where

$$A_c = \begin{bmatrix} A + BK & BK \\ \hat{A} - BK - A & \hat{A} - BK - BC \end{bmatrix}$$
$$B_{c1} = \begin{bmatrix} A_1 & 0 \\ 0 & A_1 \end{bmatrix}, \quad B_{c2} = \begin{bmatrix} B_1K & B_1K \\ 0 & 0 \end{bmatrix}$$
$$D_w = \begin{bmatrix} D \\ -D \end{bmatrix}, \quad A_R = \begin{bmatrix} A_{\rho 1} & 0 \\ A_{\rho 2} & A_{\rho 3} \end{bmatrix}$$
$$C = \begin{bmatrix} C_1 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} E_1 & 0 \end{bmatrix}$$

To adopt $\mathcal{H}_\infty$-control in the control synthesis, the following theorems will be used to establish sufficient conditions that guarantee the disturbance rejection.

*Theorem 1:* [27] The base dynamics of the closed loop system $(10)$ is asymptotically stable and $\left\|\frac{z(t)}{w(t)}\right\|_\infty \leq \gamma$, for $d, h \geq 0$ and $\gamma > 0$ if there exist positive definite matrices $\mathcal{P}$ $\mathcal{Q}_1$, and $\mathcal{Q}_2$ such that the following $ARI$ is satisfied:

$$\begin{aligned} &A_c^T\mathcal{P} + \mathcal{P}A_c + \mathcal{Q}_1 + \mathcal{Q}_2 \\ &\quad + \mathcal{P}B_{c2}\mathcal{Q}_1^{-1}B_{c2}^T\mathcal{P} + \mathcal{P}B_{c1}\mathcal{Q}_2^{-1}B_{c1}^T\mathcal{P} \\ &\quad + E^TE + \gamma^{-2}\mathcal{P}D_wD_w^T\mathcal{P} < 0 \end{aligned} \tag{11}$$

*Theorem 2:* [28] If there exists a continuously differentiable, positive definite, and radially unbounded function $V(\xi) : \mathbb{R}^n \to \mathbb{R}$ such that

$$2\dot{V}(\xi) = \frac{\partial V}{\partial x}\dot{\xi}(t) < 0 \quad x \notin \mathcal{R}$$
$$\Delta V(\xi) = V\left(A_R\xi\right) - V(\xi) \leq 0, \quad x \in \mathcal{R}$$

Then the equilibrium state is globally asymptotically stable.

## 3. Reduced Order Reset Control Systems

Consider the following special type of reset systems, when $A_{\rho 1}$ and $A_{\rho 1}$ are both zero matrices:

$$\left\{\begin{array}{c}\left[\begin{array}{c}\dot{x}_a\\ \dot{x}_b\end{array}\right]=\left[\begin{array}{cc}0 & I\\ \Lambda_1 & \Lambda_2\end{array}\right]\left[\begin{array}{c}x_a\\ x_b\end{array}\right](x_a,x_b,\tau)\notin\mathcal{R}\\ \dot{\tau}(t)=1,\\ \left[\begin{array}{c}x_a\left(t^+\right)\\ x_b\left(t^+\right)\end{array}\right]=\left[\begin{array}{cc}I & 0\\ 0 & 0\end{array}\right]\left[\begin{array}{c}x_a\\ x_b\end{array}\right](x_a,x_b,\tau)\in\mathcal{R}\\ \tau\left(t^+\right)=0,\end{array}\right. \tag{12}$$

where $x_a(t)\in\mathbb{R}^n$ represents the continuous state without reset effect, $x_b(t)\in\mathbb{R}^n$ is the resettable state, $\tau(t)>0$ and $\Lambda_i=\operatorname{diag}\left(\lambda_{i1},\lambda_{i2},\ldots,\lambda_{in}\right), i=1,2$ with $\lambda_{ij}\in\mathbb{R}$ Let $x_a(t)$ and $x_b(t)$ be partitioned as $[x_{a1}(t),\ldots,x_{an}(t)]$ and $[x_{b1}(t),\ldots,x_{bn}(t)]$, respectively. It is obvious that the system is uncoupled and becomes oscillatory if $\left(\lambda_{2i}^2+4\lambda_{1i}\right)$ is negative for all $i=1,2,\ldots,n$

*Lemma* 1 : Let the reset surface in (12) be defined as
$\mathcal{R}=\left\{\left[x_a^T(t),x_b^T(t),\tau(t)\right]^T\in\mathbb{R}^{2n+1}:\tau(t)>\bar{\tau}>0\right.$ and for any $i=0,1,\ldots,n,x_{ai}(t)=0$ such that $\left.x_{bi}(t)\neq 0\right\}$ where $\bar{\tau}=\min_{1\leq i\leq n}\{\tau_i\}$ and $\tau_i=\frac{1}{\sqrt{-\lambda_{2i}^2-4\lambda_{1i}}}$. If $\lambda_{2i}^2+4\lambda_{1i}<0$, for all $i=1,2,\ldots,n$, then the states fall exactly into the origin at most within $T=n\tau^\star$ seconds, where $\tau^\star=\max_{1\leq i\leq n}\{3\tau_i\}$

*Proof* 1 : Since $x_a(t)$ and $x_b(t)$ are completely decoupled, then every pair of $x_{ai}$ and $x_{bi}$ represents a second order subsystem for all $i=1,2,\ldots,n$. Obviously, from linear systems theory, each subsystem has complex eigenvalues since $\left(\lambda_{2i}^2+4\lambda_{1i}\right)$ is negative, with the angular frequency $\omega_i=\sqrt{-\lambda_{2i}^2-4\lambda_{1i}/2}$. Regardless of the real part sign of the eigenvalues, this oscillation guarantees that $x_{ai}(t)$ crosses the zero before time reaches the half-period $\frac{\pi}{\omega_i}$ and then $x_{bi}(t)$ becomes zero after the reset

If a certain component of $x_{ai}(t)$ crosses the zero, then $x_{bi}(t)$ vanishes due to the activation of its reset. Consequently, both $x_{ai}(t)$ and $x_{bi}(t)$ stay at the origin even if other states have not reached the equilibrium point because the system is completely decoupled. However, $x_{ai}(t)=0$ can not activate the resetting because $x_{bi}(t)=0$ as stated in $\mathcal{R}$. Every subsystem behaves in a similar manner but with different settling time depending on its angular frequency and initial conditions. Finally, the maximum time required to reset a state is less than or equal to the half period $\frac{\pi}{w_i}$ and the regulation time $\bar{\tau}$, i.e. $T_i=\frac{\pi}{w_i}+\bar{\tau}$. With straight forward calculations, it is easy to show that $T=n\tau^\star$ is the maximum time required to guarantee all trajectories settled at the origin. This completes the proof.

Now, let $A_{\rho 1}=0$ and $A_{\rho 2}=\operatorname{diag}\left(1-\delta_{1i},1-\delta_{2i},\ldots,1-\ \delta_{ni}\right)$, where $\delta_{ji}$ is the Kronecker-delta function with $\delta_{ji}=1$ if $j=i$, and zero otherwise. This structure of $A_{\rho 2}$ is used to activate resetting only for the state that hits the reset surface.

*Lemma 2:* Let the reset surface in (12) be defined as
$\mathcal{R}=\left\{\left[x_a^T(t),x_b^T(t),\tau(t)\right]^T\in\mathbb{R}^{2n+1}:\tau(t)>\bar{\tau}\right.$ and for any $i=0,1,\ldots,n,x_{bi}(t)=\alpha_i x_{ai}(t)$ such that $\left.x_{bi}(t)\neq 0\right\}$ where, $\bar{\tau}=\min_{1\leq i\leq n}\{\tau_i\}$ and $\tau_i=\frac{1}{\sqrt{-\lambda_{2i}^2-4\lambda_{1i}}}$. If there exists $s$ such that

$$\begin{gathered}\left|e^{\beta_i s}\cos\left(\omega_i s+\tan^{-1}\left(\frac{\beta_i}{\omega_i}\right)\right)\right|<\sqrt{\left(\frac{\beta_i}{\omega_i}\right)^2+1}\\ \alpha_i=\frac{x_{bi}(s)}{x_{ai}(s)},\quad i=1,2,\ldots,n\end{gathered} \tag{13}$$

and

$$\lambda_{2i}^2+4\lambda_{1i}<0$$

where, $\beta_i=\frac{\lambda_{2i}}{2}$ and $\omega_i=\sqrt{-\lambda_{2i}^2-4\lambda_{1i}/2}$, then the close loop system (3) is asymptotically stable.

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*Proof 2:* As mentioned previously, every pair $x_{ai}(t)$ and $x_{ai}(t)$ of second-order-decoupled-subsystem becomes oscillatory if $\left(\lambda_{2i}^2 + 4\lambda_{1i}\right)$ is negative for all $i = 1, 2, \ldots, n$. Then the general solution of $x_{ai}(t)$ can be written in the following form:

$$x_{ai}(t) = C_1 e^{\beta_i t} \cos\left(\omega_i t + C_2\right) \tag{14}$$

Without loss of generality, since the continuous flow has linear time invariant dynamics between reset instants, the initial time $t_0$ is assumed to be zero. Moreover, the initial condition of the resettable state can be set to zero, i.e. $x_{bi}\left(t_0\right) = 0$ for $i = 1, \ldots, n$. Now it is easy to show that the constants $C_1$ and $C_2$ are given by $\omega_i x_{ai}(0)/\sqrt{\beta_i^2 + \omega_i^2}$ and $\tan^{-1}\left(\beta_i/\omega_i\right)$ respectively.

Since $\lambda_{2i}^2 + 4\lambda_{1i} < 0$, the solution is oscillatory. Hence, the trajectory $x_{ai}(t)$ crosses the line $x_{bi}(t) = \alpha_i x_{ai}(t)$ within a finite time $s$, for any $\alpha_i$. When this occurs, $x_{bi}(t)$ vanishes because $x_{bi}\left(s^+\right) = \left(1 - \delta_{ii}\right) x_{bi}(s) = 0$ while $x_{aj}\left(s^+\right) = x_{aj}(s), j \neq i$. To ensure the stability of the trajectories then the following condition must be satisfied:

$$\left|x_{ai}\left(s^+\right)\right| = \left|x_{ai}(s)\right| < \left|x_{ai}(0)\right| \tag{15}$$

The stability of the system is ensured because Equation (15) is equivalent to $\left|x_{ai}\left(t_{k+1}\right)\right| < \left|x_{ai}\left(t_k\right)\right|$, where $t_{k+1} - t_k = s$. Substitute the solution in Equation 14 at time $t = s$ in Equation (15) gives

$$\left|\frac{\omega_i x_{ai}(0)}{\sqrt{\beta_i^2 + \omega_i^2}} e^{\beta_i t} \cos\left(\omega_i t + \tan^{-1}\left(\beta_i/\omega_i\right)\right)\right| < \left|x_{ai}(0)\right|$$

Straight forward calculations shows that this condition is equivalent to Equation (13). This shows that the point $s$ is independent on the initial condition $x_{ai}(0)$ but depends on the dynamics of the system $\beta_i$ and $\omega_i$. Then $\alpha_i$ is given at $t = s$ using $\alpha_i = \frac{x_{bi}(s)}{x_{ai}(s)}$ and contraction of the states is guaranteed. This completes the proof.

*Remark 1:* It is worth mentioning that the reset surface in Lemma 2 uses linear equation with a finite slope $\alpha_i$ rather than the zero crossing method used in Lemma 1 with infinite slope. This finite slope imposes a new condition that the decoupled components should have a separate reset rule from the other ones, i.e. the controller states are not reset at the same instant, but each one resets when its corresponding condition is satisfied. This is guaranteed by the structure of the reset matrix $A_{\rho 2}$ which is defined based on the Kronecker-delta function.

Fig. 1 demonstrates how the state $x_{ai}(t)$ is contracted by an appropriate resetting surface. It is clear that the reset surface is linear equation and its slope value is very crucial. In addition, the reset system is still stable even if the trajectory reverses its direction. Hence, the linear equation of the reset surface $x_{bi}(t) = \alpha_i x_{ai}(t)$ and $x_{bi}(t) = -\alpha_i x_{ai}(t)$ are equivalent in terms of stability. Consequently, both of these linear equations can be constructed in the reset surface $\mathcal{R}$ to guarantee fast response. The fast response occurs due to the increase of resets from both sides of the trajectories.

Now, we are in a position to generalize the results to the more general case. Consider the closed loop reset system (3) to investigate the reduced order reset controller. The following theorem provides sufficient condition for asymptotic stability of the control system (3) based on a reset controller with an order reduction.

*Theorem 3* : Let $A_{\rho 1}$ and $A_{\rho 2}$ be zero matrices. Let $\lambda_{ij} \in \mathbb{R}$ if there exist matrices $\Lambda_i = \operatorname{diag}\left(\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{im}\right), i = 1, 2, K, A_{r1}, A_{r2}, \mathcal{X}$ and $L$ with appropriate dimensions, and an invert-

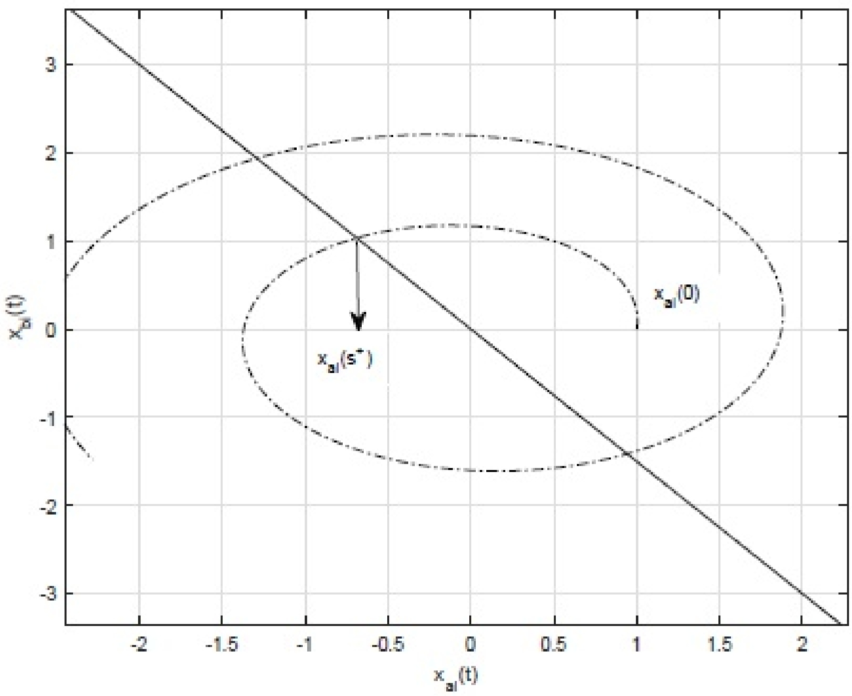


**Fig 1.** Unstable base system with the reset surface R presented in Lemma 1.

ible matrix $T$ satisfying the following equations:

$$T(A + B_p K) T^{-1} = \begin{bmatrix} 0 & 0 \\ \mathcal{X} & A_{\text{stable}} \end{bmatrix} \quad (16)$$

$$T B_p L = \begin{bmatrix} I_m \\ 0 \end{bmatrix} \quad (17)$$

$$A_{r1} T^{-1} = \begin{bmatrix} \Lambda_1 & 0 \end{bmatrix} \quad (18)$$

$$A_{r2} = \Lambda_2 \quad (19)$$

$$\Lambda_2^2 + 4\Lambda_1 < 0 \quad (20)$$

then the reset control system (3) is asymptotically stabilized using the reset surface $\mathcal{R}$ defined in Lemma 1.

*Proof 3:* Applying a similarity transformation $T_1 = \begin{bmatrix} T & 0 \\ 0 & I_m \end{bmatrix}$, with $z(t) = T_1 x(t)$ to the reset systems in (3). The instantaneous change of the state at reset instances does not change under the similarity transformation $T_1$ :

$$\begin{aligned} z(t^+) &= \begin{bmatrix} T & 0 \\ 0 & I_m \end{bmatrix} x(t^+) \\ &= \begin{bmatrix} T & 0 \\ 0 & I_m \end{bmatrix} A x(t) \\ &= \begin{bmatrix} T & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & I_m \end{bmatrix}^{-1} z(t) \\ &= \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} z(t) \end{aligned} \quad (21)$$

which is equivalent to the original jump dynamics. Using the same transformation, the continuous dynamics becomes

$$\dot{z}(t)=\begin{bmatrix} T(A+B_pK)T^{-1} & TB_pL \\ A_{r1}T^{-1} & A_{r2} \end{bmatrix} z(t) \tag{22}$$

Using Equations (16), (17), (18) and (19), Equation (22) can be written as:

$$\dot{z}(t)=\begin{bmatrix} 0 & 0 & I_m \\ \mathcal{X} & A_{\text{stable}} & 0 \\ \Lambda_1 & 0 & \Lambda_2 \end{bmatrix} z(t) \tag{23}$$

Using Equation (21) and another rearranging transformation for continuous dynamics $T_2 = \begin{vmatrix} I_m & 0 & 0 \\ 0 & 0 & I_m \\ 0 & I_{n-m} & 0 \end{vmatrix}$, with $\zeta(t) = T_2 z(t)$, the continuous and the discrete dynamics of (3) becomes

$$\dot{\zeta}(t)=\begin{bmatrix} 0 & I & 0 \\ \Lambda_1 & \Lambda_2 & 0 \\ \mathcal{X} & 0 & A_{\text{stable}} \end{bmatrix} \zeta(t) \tag{24}$$

and

$$\dot{\zeta}\left(t^{+}\right)=\begin{bmatrix} I_m & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{n-m} \end{bmatrix} \zeta(t) \tag{25}$$

respectively Now, partition $\zeta(t)$ to be

$$\zeta(t)=\begin{bmatrix} \zeta_a^T(t) & \zeta_b^T(t) & \zeta_c^T(t) \end{bmatrix}^T \tag{26}$$

It is obvious that $\zeta_c(t)$ is decoupled from the dynamics of $\zeta_a(t)$ and $\zeta_b(t)$ which means that the stability of $\zeta_a(t)$ and $\zeta_b(t)$ is independent of the stability of $\zeta_c(t)$. The continuous flow of $\zeta_a(t)$ is written as

$$\dot{\zeta}_c(t) = A_{\text{stable}}\,\zeta_c(t) + \mathcal{X}\zeta_a(t) \tag{27}$$

Since $\zeta_c(t)$ in Equation (27) is not resettable as can be seen from the discrete event in (25), the asymptotic stability of $\zeta_c(t)$ is guaranteed if $\zeta_a(t)$ is asymptotically stable. And $\zeta_a(t)$ and $\zeta_b(t)$ are asymptotically stable by Lemma 1 , this implies that $\zeta_c(t)$ is asymptotically stable. This completes the proof.

*Remark* 2 : The number of resettable states can be at most equals to the rank of $B_p$ as can be concluded from equation (17), i.e. $m \le \operatorname{rank} B_p \le n$. Hence, $m$ states of the plant vanish during a finite time as described in Lemma 1.

It is easy to conclude that $TB_p$ has a left inverse because $B_p$ has full column rank while $T$ is nonsingular. From Equation (17), the following must be satisfied

$$L=\left[(TB_p)^T(TB_p)\right]^{-1}(TB_p)^T\begin{bmatrix} I_m \\ 0 \end{bmatrix} \tag{28}$$

The following steps provide a systematic method to obtain the required variables for reduced order controller using Theorem 3. First of all we determine the rank of the input matrix $B$ in order to select the reset controller order. Let the order of the reset controller be $m$. Then, choose a matrix $K$ such that the eigenvalues of $(A + B_pK)$ are m-zeros and $n - m$ stable eigenvalues. The stable eigenvalues could be chosen using any method like pole-placement or LQR method ... etc. Now we are ready to obtain the transformation $T$. Construct $A_{stable}$ using an upper triangle matrix such that it contains the same stable eigenvalues of $(A + B_pK)$. Finally, we solve the equations in Theorem (3), keeping in mind that $\Lambda_2 = A_{r2.}L$ can be calculated using Equation (28).

*Theorem 4:* Let $A_{\rho 1}$ be a zero matrix and $A_{\rho 2} = \text{diag}(1- \ \delta_{1i}, 1-\delta_{2i}, \ldots, 1-\delta_{ni})$. And the reset surface in (12) is defined as in Lemma 2. If there exists $s$ satisfies

$$\left| e^{\beta_i s} \cos\left(\omega_i s + \tan^{-1}\left(\frac{\beta_i}{\omega_i}\right)\right)\right| < \sqrt{\left(\frac{\beta_i}{\omega_i}\right)^2 + 1} \tag{29}$$

with

$$\alpha_i = \frac{x_{bi}(s)}{x_{ai}(s)}, \quad i = 1, 2, \ldots, n$$

and there exist matrices $\Lambda_i = \text{diag}(\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{im}), i = 1, 2$ with $\lambda_{ij} \in \mathbb{R}, \mathcal{X}, L$ and an invertible matrix $T$ such that the following equations are satisfied

$$\lambda_{2i}^2 + 4\lambda_{1i} < 0$$

$$T(A + B_pK)T^{-1} = \begin{bmatrix} 0 & 0 \\ \mathcal{X} & A_{\text{stable}} \end{bmatrix} \tag{30}$$

$$TB_pL = \begin{bmatrix} I_m \\ 0 \end{bmatrix} \tag{31}$$

$$A_{r1}T^{-1} = \begin{bmatrix} \Lambda_1 & 0 \end{bmatrix} \tag{32}$$

$$A_{r2} = \Lambda_2 \tag{33}$$

$$\Lambda_2^2 + 4\Lambda_1 < 0, \tag{34}$$

where, $\beta_i = \frac{\lambda_{2i}}{2}$, then the reset control system (3)is asymptotically stable.

*Proof 4:* Similar to the proof of Theorem 3, but instead of using Lemma 1 we use Lemma 2 .

## 4. Observer-Based Reset Control Systems with Delays

The selection of the Lyapunov functional is essential to derive non-conservative sufficient conditions. Utilizing the techniques in the field of time-delay systems with the aid of Proposition (1), gives the following Theorem.

*Theorem 5:* The closed loop reset system 10 is asymptotically stable for $d, h \geq 0$ with $w(t) = 0$, if there exist positive definite matrices $\mathcal{P}$, $\mathcal{Q}_1$, and $\mathcal{Q}_2$ satisfying the following LMIs:

$$Z = \begin{bmatrix} \mathcal{P}A_c + A_c^T\mathcal{P} + \mathcal{Q}_1 + \mathcal{Q}_2 & \mathcal{P}B_{c2} & \mathcal{P}B_c1 \\ \bullet & -\mathcal{Q}_1 & 0 \\ \bullet & \bullet & -\mathcal{Q}_2 \end{bmatrix} < 0 \tag{35}$$

$$W = A_R^T\mathcal{P}A_R - \mathcal{P} < 0 \tag{36}$$

*Proof 5:* Define the Lyapunov-Krasovskii functional

$$\begin{aligned} V(\xi_t) = \xi^T(t)\mathcal{P}\xi(t) + \int_{t-d}^{t} \xi^T(s)\mathcal{Q}_1\xi^T(s)ds \\ + \int_{t-h}^{t} \xi^T(s)\mathcal{Q}_2\xi^T(s)ds \end{aligned} \tag{37}$$

with positive definite matrices $\mathcal{P}$, $\mathcal{Q}_1$ and $\mathcal{Q}_2$. Evaluating the incremental of the change of this function between the reset instance gives:

$$\begin{aligned}\Delta V(\xi_t) &= \xi^T\left(t^+\right)\mathcal{P}\xi^T\left(t^+\right) - \xi^T(t)\mathcal{P}\xi^T(t)\\ &= \xi^T(t)\left(A_R^T\mathcal{P}A_R - \mathcal{P}\right)\xi^T(t)\\ &= \xi^T(t)W\xi^T(t)\end{aligned} \tag{38}$$

which is guaranteed to be negative if the LMI in (36) is satisfied. Proceeding further, the time derivative of (37) along the solution of (10) results in:

$$\begin{aligned}\dot{V}(\xi_t) = {} & \xi^T(t)\left(\mathcal{P}A_c + A_c^T\mathcal{P} + \mathcal{Q}_1 + \mathcal{Q}_2\right)\xi^T(t)\\ & + \xi^T(t)\mathcal{P}A_1\xi^T(t-d) + \xi^T(t)\mathcal{P}B_{c2}\xi^T(t-h)\\ & + \xi^T(t-d)A_1^T\mathcal{P}\xi^T(t) + \xi^T(t-h)B_{c2}^T\mathcal{P}\xi^T(t)\\ & - \xi^T(t-d)\mathcal{Q}_1\xi^T(t-d)\\ & + \xi^T(t-h)\mathcal{Q}_2\xi^T(t-h)\\ & = \eta^T(t)Z\eta(t)\end{aligned} \tag{39}$$

where the augmented state vector $\eta(t)$ is given by $\eta^T(t) = \left[\xi^T(t) \quad \xi^T(t-d) \quad \xi^T(t-h)\right]$ and $Z$ is given by (35). The requirements of Theorem (2) are guaranteed by the negative definiteness of $Z$ and $W$. This completes the proof.

The following theorem provides LMI-procedure for stabilization of (8) using state feedback $u(t) = Kx(t)$.

*Theorem 6:* The closed loop system (8) is asymptotically stable and $\left\|\frac{z(t)}{w(t)}\right\|_\infty \leq \gamma, \gamma > 0$ for $d, h \geq 0$ if there exist positive definite matrices, $\mathcal{Q}_2$, $\mathcal{Q}_1$, and $\mathcal{S}$ and an appropriate matrix $\mathcal{Y}$, satisfying the following LMIs:

$$\begin{bmatrix} \begin{matrix}\mathcal{S}A^T + A\mathcal{S} + B\mathcal{Y}\\ +\mathcal{Y}^TB^T + \mathcal{Q}_1 + \mathcal{Q}_2\end{matrix} & \mathcal{S}E_1^T & B_1\mathcal{Y} & D & A_1\mathcal{S}\\ \bullet & -I & 0 & 0 & 0\\ \bullet & \bullet & -\mathcal{Q}_h & 0 & 0\\ \bullet & \bullet & \bullet & -\gamma^2 I & 0\\ \bullet & \bullet & \bullet & \bullet & -\mathcal{Q}_d\end{bmatrix} < 0 \tag{40}$$

$$\begin{bmatrix} -\mathcal{S} & \mathcal{S}A_{\rho 1}^T\\ \bullet & -\mathcal{S}\end{bmatrix} < 0 \tag{41}$$

Moreover, the state feedback controller is given by

$$u(t) = \mathcal{S}^{-1}\mathcal{Y}x(t) \tag{42}$$

*Proof 6:* Based on Theorems (1), (2) and (5), there exists a state feedback gain $K$ such that the closed loop system (8) is asymptotically stable if the following inequalities are satisfied:

$$\begin{aligned}& A^T\mathcal{P} + \mathcal{P}A + \mathcal{Q}_1 + \mathcal{Q}_2\\ &\quad + \mathcal{P}B_1\mathcal{Q}_1^{-1}B_1^T\mathcal{P} + \mathcal{P}A_1\mathcal{Q}_2^{-1}A_1^T\mathcal{P}\\ &\quad + E_1^TE_1 + \gamma^{-2}\mathcal{P}DD^T\mathcal{P} < 0\end{aligned} \tag{43}$$

$$A_{\rho 1}^T\mathcal{P}A_{\rho 1} - \mathcal{P} < 0 \tag{44}$$

for some positive definite matrix $\mathcal{P}$. Now, setting $\mathcal{S} = \mathcal{P}^{-1}$, $\mathcal{Y} = K\mathcal{S}$, $\mathcal{Q}_1 = \mathcal{Q}_d$, and $\mathcal{Q}_2 = \mathcal{Q}_h$, and multiplying both sides of (43) and (44) by $\mathcal{S}$, gives:

$$\begin{aligned} &\mathcal{S}A^T + A\mathcal{S} + \mathcal{Q}_d + \mathcal{Q}_h \\ &\quad + B_1\mathcal{S}\mathcal{Q}_1^{-1}\mathcal{S}B_1^T + A_1\mathcal{S}\mathcal{Q}_2^{-1}A_1^T\mathcal{S} \\ &\quad + \mathcal{S}E_1^TE_1\mathcal{S} + \gamma^{-2}DD^T < 0 \end{aligned} \tag{45}$$

$$\mathcal{S}A_{\rho 1}^T\mathcal{S}^{-1}A_{\rho 1}\mathcal{S} - \mathcal{S} < 0 \tag{46}$$

Using Schur complements, LMIs (45) and (46) are equivalent to LMIs (40) and (41), respectively. This completes the proof.

An alternative dynamic state feedback is now presented using an observer based control paradigm.

*Theorem 7:* The close loop reset-observer based control system (10) is asymptotically stable and $\left\|\frac{z(t)}{w(t)}\right\|_\infty \leq \gamma, \gamma > 0$ for $d, h \geq 0$ if there exist positive definite matrices $\mathcal{X}$, $\mathcal{Q}_{h1}$ $\mathcal{Q}_{h2}$, $\mathcal{Q}_{d1}$, $\mathcal{Q}_{d2}$ and $\mathcal{S}$ and an appropriate matrix $\mathcal{Y}$, satisfying the following LMIs:

$$\begin{bmatrix} \Theta_1 & A_1\mathcal{S} & \bar{B}_1\mathcal{Y} & \mathcal{R} \\ \bullet & -\mathcal{Q}_{d1} & 0 & 0 \\ \bullet & \bullet & -\underline{\mathcal{Q}}_{d2} & 0 \\ \bullet & \bullet & \bullet & -J \end{bmatrix} < 0 \tag{47}$$

$$\begin{bmatrix} \Theta_2 & \mathcal{X}\mathcal{N} & A_1\mathcal{X} & \bar{D} \\ \bullet & -I & 0 & 0 \\ \bullet & \bullet & -\mathcal{Q}_{ll} & 0 \\ \bullet & \bullet & \bullet & -U \end{bmatrix} < 0 \tag{48}$$

$$\begin{bmatrix} -\mathcal{S} & 0 & A_{\rho 1}^T & A_{\rho 2}^T \\ \bullet & -\mathcal{X} & 0 & A_{\rho 3}^T \\ \bullet & \bullet & -\mathcal{S} & 0 \\ \bullet & \bullet & \bullet & -\mathcal{X} \end{bmatrix} < 0 \tag{49}$$

where

$$\Theta_1 = A\mathcal{S} + \mathcal{S}A^T + B\mathcal{Y} + \mathcal{Y}^TB^T + \mathcal{Q}_{d1} + \mathcal{Q}_{h2}$$
$$\Theta_2 = A_e\mathcal{X} + \mathcal{X}A_e^T + \mathcal{Q}_{ll} + \mathcal{Q}d1$$
$$A_e = A_0 + \gamma^{-2}DD^TS^{-1}$$
$$\bar{B}_1 = [B_1B_1], \mathcal{Q}_{d2} = [\mathcal{Q}_{d2}\mathcal{Q}_{h2}]\, \bar{D} = \left[DC^TM^T\right] NN^T =$$
$$\left[\phi I - \mathcal{S}^{-1}(B\mathcal{Y} + \mathcal{Y}^TB^Tt)\mathcal{S}^{-1}\right] \text{ and } U = \left[\gamma^2 I\right].$$

Moreover, the base dynamics of the observer-based reset-controller is obtained as follows

$$\begin{cases} \dot{\xi}(t) = \left(A + B\mathcal{Y}\mathcal{S}^{-1} + \gamma^{-2}DD^T\mathcal{S}^{-1} \right. & \\ \left. -\mathcal{X}\mathcal{S}^{-T}\mathcal{Y}^TB^TD_w\right)\xi(t) + \mathcal{R}[y(t) - C\xi(t)] & \\ +B_{c1}\xi(t-d) + B_1\mathcal{Y}\mathcal{S}^{-1}\xi(t-h), & \xi(t) \notin \mathcal{R} \\ \xi\left(t^+\right) = A_R\xi(t) & x(t) \in \mathcal{R} \end{cases}$$

*Proof 7:* The proof is similar to the proof of Theorem 6.

## 5. Simulation Results

In this section, a second order linear system is used to illustrate that the presented strategy in this note is effective. Firstly, we consider a mass spring system in order to show how the reduced order reset control mechanism can be implemented. Let the spring-constant and damping-parameter values be $k = 1$ and $f_v$, respectively. The state space dynamics of the mass-spring systems becomes:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} F(t) \tag{50}$$

Since the rank of $B_p$ is unity, the reset controller can be designed with first order dynamics. According step 2, we use the pole placement method to obtain a state feedback gain $K = [-1, -2]$ in order to get $(0, -1)$ eigenvalues. Solving the equations in Theorem (3) using Procedure 1 gives $L = -1, A_{\text{stable}} = -1$. Note that $\mathcal{X}$ is arbitrary, the similarity transformation $T$ is given by

$$T = \begin{bmatrix} -0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} \tag{51}$$

when $\mathcal{X} = 1$. Let $\lambda_1 = -10$ and $\lambda_2 = 2$ are chosen according to Lemma (1), so we can obtain the controller parameters $A_{r1} = \begin{bmatrix} 5 & 5 \end{bmatrix}$ and $A_{r2} = 2$. Then the angular frequency $\omega = 2$ that restricts the regulation time to be less than $0.785$ seconds.

Fig. 2 displays the state response of the reset control system in the transformed form (24) with regulated parameter $\tau = 0.7$. It is obvious that the controller state $\zeta_b(t)$ is reset to zero when the other state $\zeta_a(t)$ vanishes. Moreover, the decoupled state $\zeta_c(t)$ is slower than the state $\zeta_a(t)$ that is intimately related to the reset state $\zeta_b(t)$. The settling time of the decoupled state can be decreased by changing its closed loop eigenvalues, i.e. eigenvalues of $A_{\text{stable}}$.

For a larger value of $\bar{\tau}$ the system will postpone the resetting event which might cause a non-acceptable increase in the state magnitude as can be seen in Fig. 3 when $\bar{\tau} = 2$. It is apparent that the zero crossing occurs before the first second but the reset controller ignores it since the time progress does not exceed the maximum of the regulated variable. This situation is demonstrated using Fig. 4 where the focus continues to cross the zero state but this cause an increase in the state magnitude which must be eliminated. The response of the Transformed reset control system is shown in Fig. 5.

For the full order reset control system with time delay (8), consider the following matrices:

$$A = \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
$$B_1 = \begin{bmatrix} 10 \\ 0-1 \end{bmatrix}, \quad D_w = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 \end{bmatrix}$$
$$A_{\rho 1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Fig. 6 illustrate the response of the dynamical system when the reset controller is not activated. It can be seen that the system is stable. Its response is slower than the response when the reset mechanism is activated as demonstrated in Fig. 7.

## 6. Conclusion

In this paper, the synthesis of the time delay reset systems scheme has been investigated. The reduced-order control method uses state feedback to change the eigenvalues and then the reset controller generates a new set of complex eigenvalues to guarantee finite-time convergence. It is demonstrated that unstable base systems can be stabilized using reset mechanisms with very fast responses.

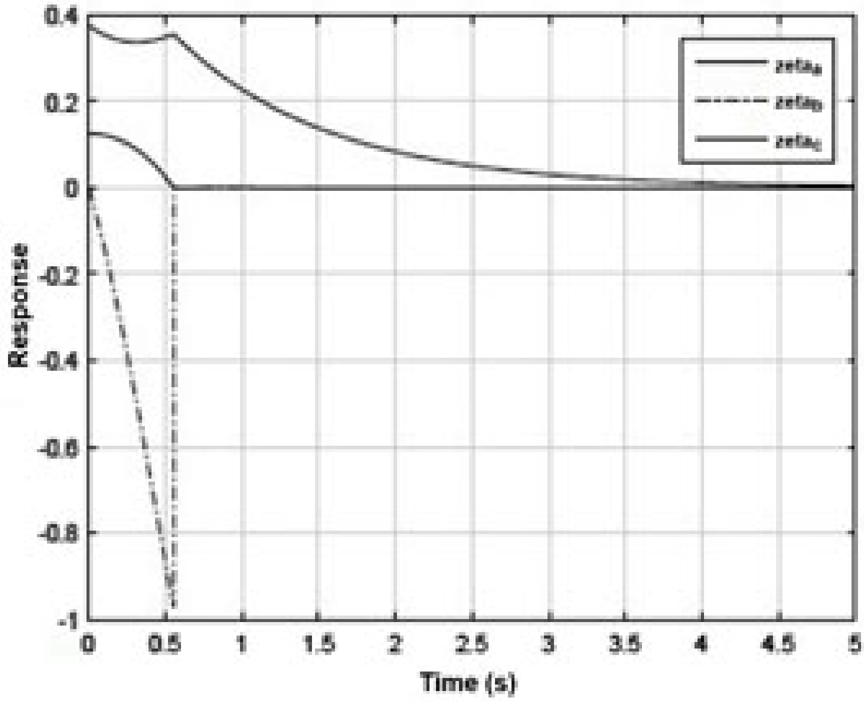


**Fig 2.** State response of the reset control system in the transformed form when $\bar{\tau}$ = 0.7

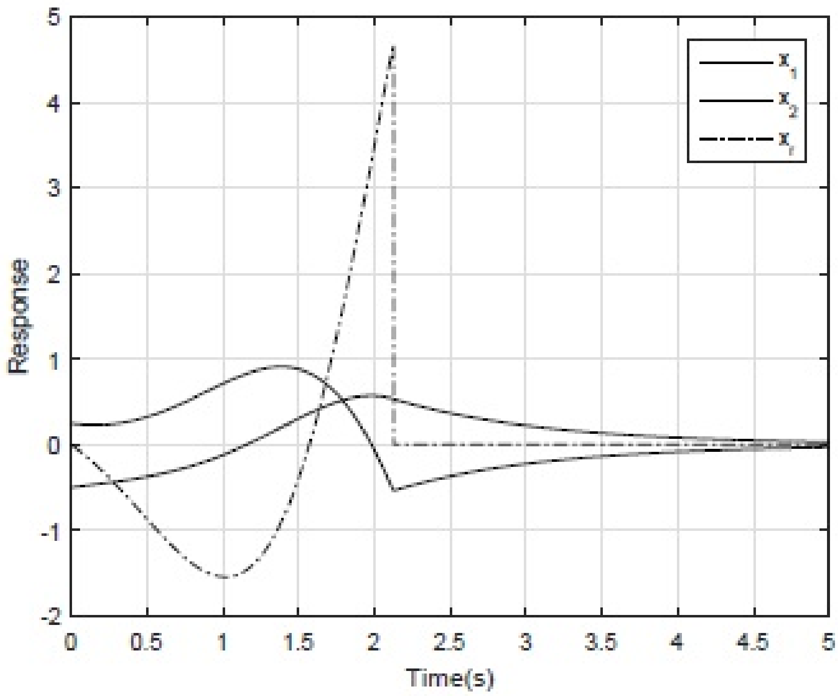


**Fig 3.** State response of the reset control system in the original form when $\bar{\tau}$ =2

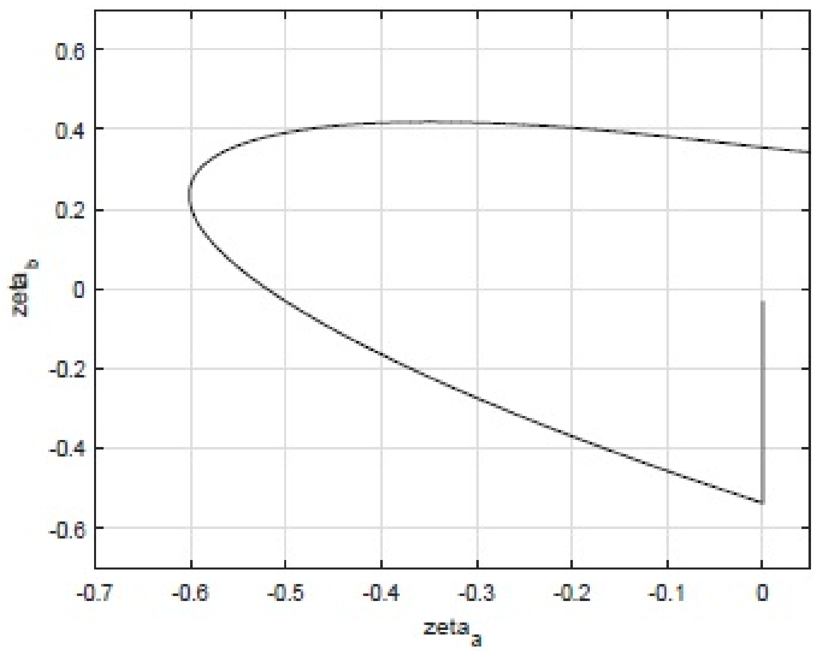


**Fig 4.** Oscillation behavior of the designed reset system when $\bar{\tau}$ =2

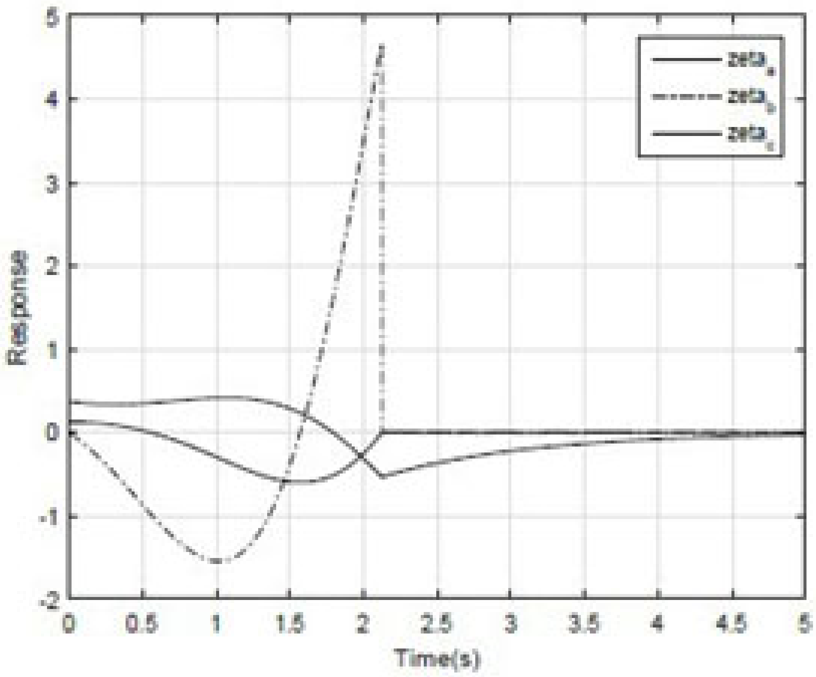


**Fig 5.** State response of the reset control system in the transformed form when $\bar{\tau}$ =2

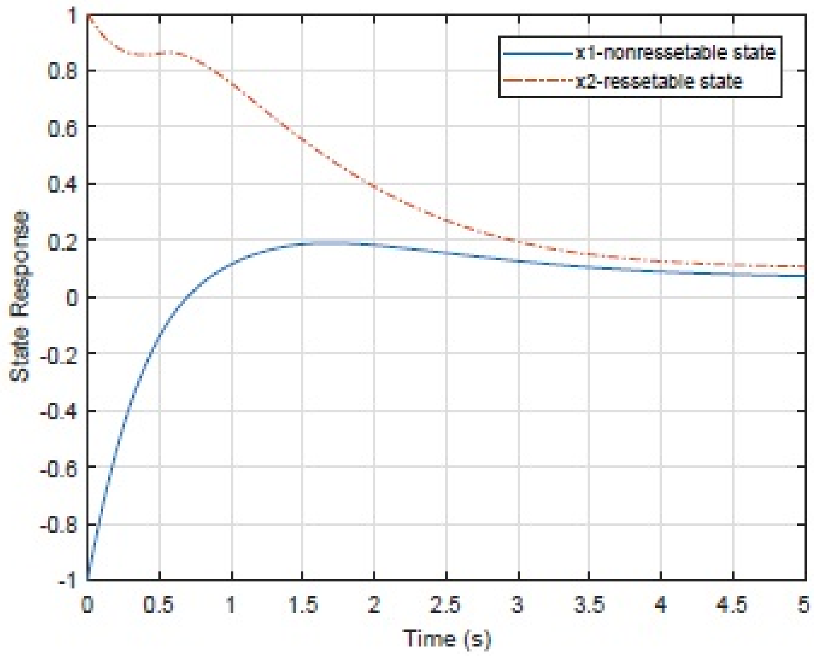


**Fig 6.** The trajectory response of the system without the reset methodology

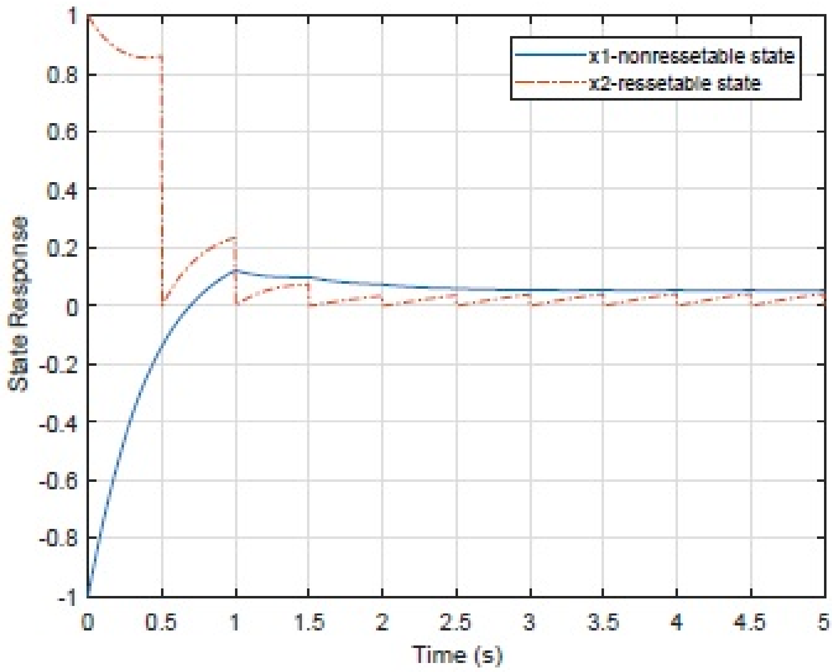


**Fig 7.** The trajectory response of the system with the reset mechanism

The methodologies allow users to decouple reset controllers for a set of states and the remaining states can be designed by classical methods. Based on linear matrix inequalities, we extend the results to obtain controller gains for time-delay systems. The simulation example demonstrates the effectiveness of the proposed theoretical results to stabilize unstable systems using reset controllers.

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