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Fractional-Order Discrete Predator–Prey System of Leslie Type: Existence, Stability, and Numerical Simulation

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ABSTRACT

This study explores a fractional-order (FO) discrete predator prey (PP) system of Leslie type (LT) by incorporating fractional differences in the Caputo-Fabrizio-Riemann (CFR) sense. We rigorously establish the existence and uniqueness of solutions and provide a comprehensive stability analysis. A novel numerical scheme is developed to approximate the system's dynamics, yielding deeper insights into PP interactions under FO effects. Furthermore, we validate our theoretical findings using numerical simulations, which confirm the robustness and accuracy of the proposed model. The results underline the significance of fractional calculus (FC) in ecological modeling and pave the way for future investigations in population dynamics.

1. Introduction

FC, which generalizes differentiation and integration to non-integer orders, excels in modeling systems with memory effects due to its integral-based operators that encode historical dependencies. This makes FC particularly effective in fields like viscoelasticity, anomalous diffusion, and wave propagation, where power-law memory dynamics dominate. Recent studies highlight its potential in emerging areas enhancing chaotic dynamical systems (e.g., Halvorsen systems) under white noise perturbations, improving global energy utilization projections through memory-aware machine learning frameworks, and optimizing satellite image classification by capturing long-range spatial-temporal correlations via vision transformers.

Additionally, FC could refine statistical modeling in geometric processes with Maxwell distributions, offering superior accuracy over classical methods [59]–[62]. These capabilities underscore FC's potential for enhancing ecological modeling frameworks. Traditional PP systems, long serving as fundamental models in ecological studies, could particularly benefit from FC's memory-encoding operators. While classical PP models [1]–[5] effectively describe basic species interactions and population fluctuations, they often overlook historical dependencies in population dynamics and environmental feedback loops. FC operators could enable more nuanced representations of delayed trophic interactions, cumulative resource depletion effects, and trans-generational population memory This integration would extend the classical Lotka-Volterra paradigm by incorporating temporal non-locality through fractional-order operators, potentially improving predictions of complex ecological

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phenomena like regime shifts and extinction cascades.

Ecological dynamics, such as the well-known lynx–hare cycles, underscore the profound implications of PP interactions on ecosystem stability and biodiversity conservation. Traditional discrete models, such as the LT system, have significantly contributed to our understanding of these dynamics; however, they often neglect the memory and hereditary effects that play a vital role in biological processes [7], [11].

This limitation poses a significant problem because memory effects—such as delayed predator responses to sudden prey booms—can critically alter the amplitude and period of population cycles, leading to inaccurate predictions in traditional models. FO PP models naturally embed these nonlocal temporal dependencies, offering more realistic and robust predictions of ecosystem dynamics under fluctuating environmental conditions [9], [11], [14], [25]–[27].

Recent investigations have demonstrated that these memory effects, can have profound implications on system behavior [9], [14]. Moreover, similar FO discrete approaches have been successfully applied in other complex biological models, such as reaction–diffusion systems [15]–[20] and epidemic models, underscoring the versatility of fractional discrete calculus (FDC) in capturing intricate dynamics.

While FO models offer enhanced capabilities in capturing memory effects within ecological systems, many classical models fail to adequately incorporate the long-term memory inherent in these processes—resulting in inaccurate predictions of population fluctuations, delayed oscillations, and chaotic dynamics. Moreover, FO models introduce challenges such as increased model complexity, higher computational demands, and difficulties in parameter estimation, which further complicate the interpretation of results. Recognizing these issues, our study employs the CFR fractional difference framework, which utilizes a non-singular kernel to enhance both computational efficiency and model flexibility, thereby facilitating a more nuanced representation of memory effects in PP interactions and significantly improving the model's predictive capabilities and applicability to diverse ecological scenarios. However, unsolved problems remain, including the trade-off between model accuracy and computational feasibility, and the lack of efficient frameworks that balance these aspects. Existing FO models often sacrifice simplicity for precision or become impractical due to computational demands, limiting their widespread use. In light of these limitations, FO has emerged as a powerful tool for modeling long-term memory and non-local interactions. Our study addresses these gaps by integrating insights from existing literature to develop a more balanced and efficient modeling approach, thereby advancing the understanding of PP dynamics within the framework of FO calculus. Early work laid the foundation for the application of fractional order derivatives (FOD) in continuous systems, while more recent studies have successfully extended these concepts to discrete frameworks. Unlike prior discrete PP studies, our CFR framework captures memory-dependent oscillations and enables the derivation of sharper stability thresholds, thereby providing more accurate predictions of species dynamics under environmental stress. Similarly, Sun and Chen provided comprehensive insights into the stability and chaotic behavior of discrete systems governed by FO difference equations. This study aims to answer how can fractional differences in the CFR sense be integrated into discrete PP models to capture memory effects efficiently while overcoming existing limitations.This question is significant as it seeks to improve ecological modeling for better ecosystem management. [22]–[25], [46]

Building on these advancements, Li, Zhao, and Chen developed numerical methods specifically tailored for FDE in ecological models, thereby improving the accuracy of numerical simulations. Zhao and Wang further highlighted the importance of FO discrete calculus in capturing the subtle nuances of ecological interactions, Almeida and Baleanu have also contributed to the theoretical underpinnings by establishing new existence and uniqueness results for discrete FO systems. Recent research has additionally focused on the stability analysis of such models. Chen and Li proposed a novel numerical scheme for FDE in PP dynamics, while Li and Zeng rigorously examined the conditions for the existence and stability of solutions in these systems [26]–[28], [30], [31], [36]–[39]. Further robust frameworks for analyzing uniform stability have been developed by Xu and Guo, with Wang providing a comprehensive review of recent advances in FO discrete dynamical systems and their applicability to ecological modeling [29], [32], [33]. Furthermore, the development of efficient numerical algorithms has been a major focus in recent years. Chen has demonstrated that tailored numerical schemes can capture complex dynamics such as quasi-periodicity and chaos, which are often observed in real-world ecological systems. Sun and Huang have also contributed by proposing new schemes that reduce computational complexity while maintaining high accuracy *et al.* [40]–[43]. In addition, Duman [44] explored the chaotic dynamics of FO discrete PP models, revealing how small changes in system parameters can lead to significant variations in population behavior. Recent trends in the field are further encapsulated by Li and Abdalla [45], [46], who reviewed emerging approaches in FO discrete calculus and their applications in biology.

In this work, we extend the classical discrete PP model by introducing a FO difference formulation based on the RL approach. The selection of the RL fractional difference framework in our model is motivated by its mathematical tractability and suitability for discrete-time systems. Compared to other definitions, such as Caputo or Caputo-Fabrizio, the RL approach offers a more straightforward implementation in the discrete domain, facilitating the analysis of system stability and the development of efficient numerical schemes. This choice aligns with our objective to model PP interactions with greater precision while maintaining computational feasibility. This enhancement provides a more nuanced description of population dynamics by accounting for historical states and their influence on current behavior. Our study rigorously demonstrates the existence and uniqueness of solutions drawing upon the stability analyses discussed in [29], [33]–[35] and further substantiated by recent numerical experiments [41], [42] and develops a tailored numerical scheme to effectively approximate the system's dynamics, building on the methodologies of [26], [27]. Our main findings demonstrate that the CFR-based scheme uncovers memory-driven oscillations that are absent in integer-order models but also delivers sharper stability thresholds for PP coexistence.

The remainder of this paper is structured as follows. Section 2 details the formulation of the FO PP model, including its derivation and underlying assumptions. Section 3 is devoted to the qualitative analysis of the model, where we establish the existence, uniqueness, and stability of solutions. Section 4 presents a quantitative analysis through numerical simulations and discussions on the dynamic behavior of the system.

2. Mathematical Model

We extend the classical discrete PP model by incorporating a FOD operator based on the RL definition. This formulation is designed to capture long-term memory and non-local interactions that are inherent in many biological processes [22], [23]. Our model builds upon recent advances in FDC [9], [24] and adapts these concepts to better describe the evolution of ecological systems. We begin by defining the RL-FDO and then develop the FO discrete PP system that serves as the foundation for our subsequent analysis.

The study of PP dynamics has a long history. Early contributions by Lotka and Volterra established the foundation of continuous models describing species interactions [49], [50]. Later, discrete models, such as those developed by Leslie [7], were introduced to capture population changes in discrete time steps and to facilitate numerical simulations. More recently, FC has been applied to extend these models by incorporating memory and non-local effects, offering a more comprehensive framework to describe complex ecological interactions [51]–[54]. Our methodological framework begins by selecting the CFR fractional difference operator over Caputo and Riemann–Liouville due to its non-singular kernel and computational tractability (Definition 2.1). We derive a FO extension of the Leslie system using Riemann–Liouville differences (Equations 1–5). The stability analysis is conducted by combining Lipschitz continuity (Lemma 3.1) with the discrete Gronwall inequality (Lemma 3.2). Validation is performed through a dual protocol involving classical recovery when $\varpi = 1$ and synthetic benchmark testing. The transition from classical to fractional modeling is justified by empirical evidence showing memory effects in PP systems, such as delayed numerical responses. Traditional models fail to capture long-range temporal correlations observed in such dynamics, whereas fractional operators naturally incorporate hysteresis through their non-local kernels.

A classical discrete PP model is governed by a system of difference equations that describe the dynamics of the prey population $\mathfrak{x}_t$ and the predator population $\mathfrak{y}_t$ over discrete time steps. A typical form of such a model is:

$$\begin{cases} \mathfrak{x}_{t+1} = \mathfrak{x}_t + \delta \mathfrak{x}_t \left[(1 - \mathfrak{x}_t) - \dfrac{\mathfrak{y}_t}{\mathfrak{x}_t^2 + c} \right], \\ \mathfrak{y}_{t+1} = \mathfrak{y}_t + \delta \mathfrak{y}_t \left[a - \dfrac{b\mathfrak{y}_t}{\mathfrak{x}_t} \right], \end{cases} \tag{1}$$

To make the link to biology explicit, we summarize the role of each parameter in Table 1.

Table 1. Ecological interpretation of model parameters

Parameter	Meaning
a	Predator growth rate coefficient
b	Predator mortality rate coefficient
c	Half-saturation constant in functional response
δ	Time-step scaling factor (applies to both equations)
$\mathfrak{x}_t$	Prey population at time t
$\mathfrak{y}_t$	Predator population at time t

Define the forward difference operator by

$$\Delta \vartheta(t) = \vartheta(t+1) - \vartheta(t),$$

For any function ϑ defined on

$$\mathbb{N}_{t_0} := \{t_0, t_0 + 1, t_0 + 2, \dots\},$$

The transition to (2) ensures a discrete-time formulation while preserving key dynamics.

$$\begin{cases} \Delta \mathfrak{x}(t) = \delta \mathfrak{x}(t) \left[(1 - \mathfrak{x}(t)) - \dfrac{\mathfrak{y}(t)}{\mathfrak{x}^2(t) + c} \right], \\ \Delta \mathfrak{y}(t) = \delta \mathfrak{y}(t) \left[a - \dfrac{b\mathfrak{y}(t)}{\mathfrak{x}(t)} \right], \end{cases} \tag{2}$$

This formulation captures the essential interaction mechanisms between the two species [7]. The shift from integer-order to FO models is motivated by several factors. First, fractional operators incorporate the influence of past states, thereby capturing the hereditary properties observed in ecological systems [22]. Moreover, these operators enable the modeling of non-local interactions, allowing for the representation of dynamics that extend over longer time periods and thus reflecting more realistic behaviors in natural ecosystems [24]. Additionally, FO models can exhibit a richer set of dynamical behaviors, including delayed responses, periodic oscillations, and chaos, which are often observed in empirical ecological studies. Finally, by accounting for historical influences, these models offer improved predictive capability over classical integer-order models, leading to a better understanding of complex population dynamics.

Definition 2.1 [47] The *Caputo–Fabrizio–Caputo (CFC) fractional difference* is given by

$$\begin{aligned}{}^{\mathrm{CFC}}{}_{t_0}\Delta^{\varpi}\vartheta(t) &= \frac{\mathcal{M}(\varpi)}{1-\varpi}\sum_{\xi=t_0}^{t-1}\left(\Delta_{\xi}\vartheta\right)(\xi)\left(1+\Lambda\right)^{t-\sigma(\xi)}\\ &= \frac{\mathcal{M}(\varpi)}{1-2\varpi}\sum_{\xi=t_0}^{t-1}\left(\Delta_{\xi}\vartheta\right)(\xi)\left(1+\Lambda\right)^{t-\xi}, \quad \varpi\in(0,1).\end{aligned} \tag{3}$$

Where $\mathcal{B}(\varpi)$ and $\Lambda = -\dfrac{\varpi}{1-\varpi}$ denotes a positive normalization constant such that $\mathcal{M}(0) = \mathcal{M}(1) = 1$.

Definition 2.2 [48] The CFR fractional difference is defined by

$$\begin{aligned}{}^{\mathrm{CFR}}{}_{t_0}\Delta^{\varpi}\vartheta(t) &= \frac{\mathcal{M}(\varpi)}{1-\varpi}\Delta_t\sum_{\xi=t_0}^{t-1}\vartheta(\xi)\left(1+\Lambda\right)^{t-\sigma(\xi)}\\ &= \frac{\mathcal{M}(\varpi)}{1-2\varpi}\Delta_t\sum_{\xi=t_0}^{t-1}\vartheta(\xi)\left(1+\Lambda\right)^{t-\xi}, \quad \varpi\in(0,1).\end{aligned} \tag{4}$$

The CFC and CFR operators were chosen for their non-singular kernels, memory-dependent weighting, and moderate computational cost. However, they require more memory for long simulations and their non-local nature complicates parallel computation. The fractional version of the proposed discrete model (2) can be given as follows:

$$\begin{cases} {}^{CFR}{}_{t_0}\Delta^{\varpi}\mathfrak{x}(t) = \delta\mathfrak{x}(t)\left[1-\mathfrak{x}(t)-\dfrac{\mathfrak{y}(t)}{\mathfrak{x}(t)^2+c}\right], \\ {}^{CFR}{}_{t_0}\Delta^{\varpi}\mathfrak{y}(t) = \delta\mathfrak{y}(t)\left[a-\dfrac{b\mathfrak{y}(t)}{\mathfrak{x}(t)}\right], \end{cases} \tag{5}$$

Where $\varpi > 0$, $t \in \mathbb{N}_{t_0} = \{t_0, t_0+1, \dots\}$ such that $t_0 \in \mathbb{R}$, and ${}^{CFR}{}_{t_0}\Delta^{\varpi}$ is the CFR fractional differences. The integer-order model (Equation 1) is extended to the FO framework by first replacing classical differences with the CFR operator (Definition 2.1). A memory kernel $\mathcal{M}(\varpi)(1+\Lambda)^{t-\xi}$ is incorporated to capture nonlocal effects. Ecological interpretability is preserved by constraining parameters, ensuring $\delta < \frac{1-\varpi}{L}$ (Theorem 3.3) for stability and $c > \mathfrak{x}_t^2$ to avoid singularities. The CFR operator offers strong stability with second-order accuracy $\mathcal{O}(h^2)$ and medium computational cost. In contrast, the CFC operator provides only first-order accuracy $\mathcal{O}(h)$ with moderate stability but benefits from low computational cost. The classical Caputo operator, while offering variable accuracy $\mathcal{O}(h^{2-\varpi})$, tends to be less stable and computationally expensive.

With the model now rigorously defined, we proceed to its qualitative analysis.We establish the existence and uniqueness of solutions by employing fixed-point theorems and the intrinsic properties of fractional difference operators. Moreover, we derive sufficient conditions for the uniform stability of the system, thereby laying the foundation for understanding the complex dynamics inherent in the FO-PP model.

3. Qualitative Analysis

In the qualitative analysis, we rigorously investigate the theoretical properties of the FO-PP model. This includes establishing the existence and uniqueness of solutions, as well as deriving sufficient conditions for the uniform stability of the system. Our analysis leverages fixed-point theorems and the inherent properties of fractional difference operators, providing a solid foundation for understanding the system's behavior from a theoretical standpoint.

3.1. Existence and Uniqueness

This section presents the theoretical analysis of existence and uniqueness for the proposed FO-PP model. Using fixed-point theorems and leveraging properties of fractional difference equations, we establish rigorous conditions that guarantee a unique solution. Our approach is inspired by recent contributions in the field [28], [32], [33], which have provided robust frameworks for ensuring well-posedness in discrete fractional systems. This analysis is critical for confirming that the model is mathematically sound and reliable for representing ecological dynamics.

Definition 3.1 [47] Let $\varpi \in (0,1)$ and let ϑ be a function defined on $\mathbb{N}_{t_0}$. The left discrete delta Caputo–Fabrizio (CF) fractional sum of order ϖ is defined by

$$ {}^{CF}{}_{t_0}\Delta^{-\varpi}\vartheta(t) = \frac{1-\varpi}{\mathcal{M}(\varpi)}\vartheta(t) + \frac{\varpi}{\mathcal{M}(\varpi)}\sum_{\xi=t_0}^{t-1}\vartheta(\xi), \quad \forall t \in \mathbb{N}_{t_0}, \tag{6}$$

Lemma 3.1 [48] Building on Definition 2.1, the limiting behavior of the operator is characterized as follows:

(i) As $\varpi \to 0$, the operator converges to the classical difference:

$$\begin{aligned} \left({}^{CFC}{}_{t_0}\Delta^{\varpi}\vartheta\right)(t) &\to \vartheta(t) - \vartheta(t_0), \\ \left({}^{CFR}{}_{t_0}\Delta^{\varpi}\vartheta\right)(t) &\to \vartheta(t). \end{aligned}$$

(ii) As $\varpi \to 1$, we have

$$\begin{aligned} \left({}^{CFC}{}_{t_0}\Delta^{\varpi}\vartheta\right) t) &\to \Delta\vartheta(t), \\ \left({}^{CFR}{}_{t_0}\Delta^{\varpi}\vartheta\right)(t) &\to \Delta\vartheta(t). \end{aligned}$$

As $\varpi \to 0$, regularization is required to avoid division-by-zero issues, and the system often becomes stiff, necessitating implicit numerical methods. As $\varpi \to 1$, the transition toward integer-order derivatives demands adaptive time-stepping, and the kernel summation can become ill-conditioned.

Lemma 3.2 [47] Let $\varpi \in (0,1)$, and $\vartheta : \mathbb{N}_{t_0} \to \mathbb{R}$ be a function. Then, the following identities hold for all $t \in \mathbb{N}_{t_0}$:

$$ {}^{CF}{}_{t_0}\Delta^{-\varpi}\, {}^{CFR}{}_{t_0}\Delta^{\varpi}\vartheta(t) = \vartheta(t), \tag{7}$$

$$ {}^{CFR}{}_{t_0}\Delta^{\varpi}\, {}^{CF}{}_{t_0}\Delta^{-\varpi}\vartheta(t) = \vartheta(t). \tag{8}$$

Theorem 3.1 The system (5) admits a unique solution given by:

$$\begin{aligned} \mathfrak{x}(t) = {} & \frac{1-\varpi}{\mathcal{M}(\varpi)}\left[\delta\mathfrak{x}(t)\left(1-\mathfrak{x}(t)-\frac{\mathfrak{y}(t)}{\mathfrak{x}(t)^2+c}\right)\right] \\ & + \frac{\varpi}{\mathcal{M}(\varpi)}\sum_{\xi=t_0}^{t-1}\left[\delta\mathfrak{x}(\xi)\left(1-\mathfrak{x}(\xi)-\frac{\mathfrak{y}(\xi)}{\mathfrak{x}(\xi)^2+c}\right)\right], \quad \forall t \in \mathbb{N}_{t_0}, \\ \mathfrak{y}(t) = {} & \frac{1-\varpi}{\mathcal{M}(\varpi)}\left[\delta\mathfrak{y}(t)\left(a-\frac{b\mathfrak{y}(t)}{\mathfrak{x}(t)}\right)\right] \\ & + \frac{\varpi}{\mathcal{M}(\varpi)}\sum_{\xi=t_0}^{t-1}\left[\delta\mathfrak{y}(\xi)\left(a-\frac{b\mathfrak{y}(\xi)}{\mathfrak{x}(\xi)}\right)\right], \quad \forall t \in \mathbb{N}_{t_0}. \end{aligned} \tag{9}$$

Proof 1 We apply the left discrete delta CF fractional sum operator ${}^{CF}{}_{t_0}\Delta^{-\varpi}$ to both sides of the FO system (5):
For the prey population $\mathfrak{x}(t)$:

$$ {}^{CF}{}_{t_0}\Delta^{-\varpi}\left({}^{CFR}{}_{t_0}\Delta^{\varpi}\mathfrak{x}(t)\right) ={}^{CF}{}_{t_0}\Delta^{-\varpi}\left[\delta\mathfrak{x}(t)\left(1-\mathfrak{x}(t)-\frac{\mathfrak{y}(t)}{\mathfrak{x}(t)^2+c}\right)\right]. \tag{10} $$

For the predator population $\mathfrak{y}(t)$:

$$ {}^{CF}{}_{t_0}\Delta^{-\varpi}\left({}^{CFR}{}_{t_0}\Delta^{\varpi}\mathfrak{y}(t)\right) = {}^{CF}{}_{t_0}\Delta^{-\varpi}\left[\delta\mathfrak{y}(t)\left(a-\frac{b\mathfrak{y}(t)}{\mathfrak{x}(t)}\right)\right]. \tag{11} $$

From Lemma 3.2, we have the identity:

$$ {}^{CF}{}_{t_0}\Delta^{-\varpi}\left({}^{CFR}{}_{t_0}\Delta^{\varpi}\mathfrak{x}(t)\right) = \mathfrak{x}(t). $$

Applying this to (10) and (11) yields:
For $\mathfrak{x}(t)$:

$$ \mathfrak{x}(t) = {}^{CF}{}_{t_0}\Delta^{-\varpi}\left[\delta\mathfrak{x}(t)\left(1-\mathfrak{x}(t)-\frac{\mathfrak{y}(t)}{\mathfrak{x}(t)^2+c}\right)\right]. \tag{12} $$

For $\mathfrak{y}(t)$:

$$ \mathfrak{y}(t) = {}^{CF}{}_{t_0}\Delta^{-\varpi}\left[\delta\mathfrak{y}(t)\left(a-\frac{b\mathfrak{y}(t)}{\mathfrak{x}(t)}\right)\right]. \tag{13} $$

Using Definition 3.1, the fractional sum operator expands as:

$$ \begin{cases} \mathfrak{x}(t) = \dfrac{1-\varpi}{\mathcal{M}(\varpi)}\mathfrak{F}(\mathfrak{x}(t),\mathfrak{y}(t)) + \dfrac{\varpi}{\mathcal{M}(\varpi)}\sum\limits_{\xi=t_0}^{t-1}\mathfrak{F}(\mathfrak{x}(\xi),\mathfrak{y}(\xi)), & \forall t\in\mathbb{N}_{t_0}, \\ \mathfrak{y}(t) = \dfrac{1-\varpi}{\mathcal{M}(\varpi)}\mathfrak{G}(\mathfrak{x}(t),\mathfrak{y}(t)) + \dfrac{\varpi}{\mathcal{M}(\varpi)}\sum\limits_{\xi=t_0}^{t-1}\mathfrak{G}(\mathfrak{x}(\xi),\mathfrak{y}(\xi)), & \forall t\in\mathbb{N}_{t_0}, \end{cases} \tag{14} $$

Where

$$ \mathfrak{F}(\mathfrak{x},\mathfrak{y}) = \delta\mathfrak{x}\left(1-\mathfrak{x}-\frac{\mathfrak{y}}{\mathfrak{x}^2+c}\right), $$
$$ \mathfrak{G}(\mathfrak{x},\mathfrak{y}) = \delta\mathfrak{y}\left(a-\frac{b\mathfrak{y}}{\mathfrak{x}}\right). $$

This completes the proof of existence. Uniqueness follows from the Lipschitz continuity of $\mathfrak{F}$ and $\mathfrak{G}$ under the given parameter constraints, ensuring the fixed-point solution is unique.

The model relies on several key assumptions. Memory decay is represented using an exponential kernel, which aligns well with ecological data . Linearity is assumed to hold for moderate population densities, and stationarity is considered valid over short-term simulations. In contrast, simpler integer-order models fail to capture essential features such as delayed predator numerical responses and non-Markovian population dynamics.

3.2. Stability Analysis

We conducted a comprehensive stability analysis of the FO discrete PP model. By employing linearization techniques and utilizing the inherent stability properties of FO difference equations, we derive sufficient conditions for uniform stability of the system. The methodology used here is grounded in recent studies that explore the dynamic behavior of FO discrete systems [29], [30], [44]. Our findings provide key insights into the resilience and long-term behavior of ecological populations under FO dynamics.

Lemma 3.3 Let $\mathfrak{F}$ and $\mathfrak{G}$ be functions defined on $\mathbb{R}^2$, and let $\delta, c, a, b > 0$ be given constants. Then, for any compact set

$$\mathcal{D} \subset \{(\mathfrak{x}, \mathfrak{y}) \in \mathbb{R}^2 : \mathfrak{x} > \eta \text{ and } \mathfrak{y} \geq 0\}, \tag{15}$$

with some $\eta > 0$, the functions $\mathfrak{F}$ and $\mathfrak{G}$ are Lipschitz continuous on $\mathcal{D}$.

Proof 2 Let $\mathcal{D}$ be a fixed compact subset of $\{(\mathfrak{x}, \mathfrak{y}) \in \mathbb{R}^2 : \mathfrak{x} > \eta,\ \mathfrak{y} \geq 0\}$ for some $\eta > 0$. We aim to show that $\mathfrak{F}$ and $\mathfrak{G}$ are Lipschitz continuous on $\mathcal{D}$ by demonstrating that their first-order partial derivatives are bounded on $\mathcal{D}$.
For $\mathfrak{F}$, the derivatives with respect to $\mathfrak{x}$ and $\mathfrak{y}$ are given respectively by

$$\partial_{\mathfrak{x}} \mathfrak{F}(\mathfrak{x}, \mathfrak{y}) = \delta \left(1 - 2\mathfrak{x} - \frac{\mathfrak{y}}{\mathfrak{x}^2 + c} + \frac{2\mathfrak{x}^2 \mathfrak{y}}{(\mathfrak{x}^2 + c)^2} \right), \tag{16}$$

$$\partial_{\mathfrak{y}} \mathfrak{F}(\mathfrak{x}, \mathfrak{y}) = -\frac{\delta \mathfrak{x}}{\mathfrak{x}^2 + c}. \tag{17}$$

Since $\mathcal{D}$ is compact and $\mathfrak{x} \geq \eta > 0$ within $\mathcal{D}$, all terms in these expressions are bounded. In particular, for the second derivative, observe that the function $f(\mathfrak{x}) = \mathfrak{x}/(\mathfrak{x}^2 + c)$ attains its maximum at $\mathfrak{x} = \sqrt{c}$, yielding

$$|\partial_{\mathfrak{y}} \mathfrak{F}(\mathfrak{x}, \mathfrak{y})| \leq \frac{\delta}{2\sqrt{c}}. \tag{18}$$

For $\mathfrak{G}$, we compute

$$\partial_{\mathfrak{x}} \mathfrak{G}(\mathfrak{x}, \mathfrak{y}) = \frac{\delta b\, \mathfrak{y}^2}{\mathfrak{x}^2}, \tag{19}$$

$$\partial_{\mathfrak{y}} \mathfrak{G}(\mathfrak{x}, \mathfrak{y}) = \delta \left(a - \frac{2b\, \mathfrak{y}}{\mathfrak{x}} \right). \tag{20}$$

Again, using the boundedness of $\mathfrak{y}$ and the fact that $\mathfrak{x} > \eta$ on $\mathcal{D}$, both expressions are bounded.

As all first-order partial derivatives of $\mathfrak{F}$ and $\mathfrak{G}$ are bounded on the compact set $\mathcal{D}$, the Mean Value Theorem [55] ensures that both functions are Lipschitz continuous on $\mathcal{D}$. Consequently, by the Picard–Lindelöf theorem [56], the corresponding initial value problems admit unique local solutions.

Lemma 3.4 Discrete Grönwall Inequality [57] Let $\{u_n\}_{n \geq t_0}$ be a nonnegative sequence satisfying

$$u_n \leq A + \sum_{k=t_0}^{n-1} b_k\, u_k, \quad \text{for all } n \geq t_0, \tag{21}$$

Where $A \geq 0$ is a constant and $\{b_k\}_{k \geq t_0}$ is a sequence of nonnegative real numbers. Then the following bound holds:

$$u_n \leq A \exp\left(\sum_{k=t_0}^{n-1} b_k \right), \quad \text{for all } n \geq t_0. \tag{22}$$

Theorem 3.2 [58] The FO version of the proposed discrete model (3) is said to be uniformly stable if, for every $\epsilon > 0$, there exist constants δ_ϵ and $T > 0$, with $0 < \delta_\epsilon < \epsilon$, such that for all $t \in \mathbb{N}_{t_0+1} = \{t_0 + 1, t_0 + 2, \ldots, T\}$, and for any two solutions $\phi(t; t_0, \vartheta)$ and $\psi(t; t_0, \theta)$ of the model with initial conditions $\phi(t_0) = \vartheta$ and $\psi(t_0) = \theta$, the condition

$$\|\theta - \vartheta\| < \delta_\epsilon \tag{23}$$

implies that

$$\|\phi(t) - \psi(t)\| < \epsilon. \tag{24}$$

Theorem 3.3 Let $\mathfrak{F}(\mathfrak{x}, \mathfrak{y})$ and $\mathfrak{G}(\mathfrak{x}, \mathfrak{y})$ be Lipschitz continuous in their arguments with respective positive constants $L_{\mathfrak{F}}$ and $L_{\mathfrak{G}}$. If the condition

$$\mathcal{M}(\varpi) > (1-\varpi)L, \quad \text{where} \quad L = \max\{L_{\mathfrak{F}}, L_{\mathfrak{G}}\} \tag{25}$$

holds, then the system (5) is uniformly stable.

Proof 3 Let $\phi(t) = (\mathfrak{x}_1(t), \mathfrak{y}_1(t))$ and $\psi(t) = (\mathfrak{x}_2(t), \mathfrak{y}_2(t))$ be two solutions of (5) with initial conditions $\phi(t_0) = \vartheta$ and $\psi(t_0) = \theta$, respectively.
From Theorem 3.1, the solutions satisfy the integral equations:

$$\phi(t) = \frac{1-\varpi}{\mathcal{M}(\varpi)} \begin{bmatrix} \mathfrak{F}(\mathfrak{x}_1(t), \mathfrak{y}_1(t)) \\ \mathfrak{G}(\mathfrak{x}_1(t), \mathfrak{y}_1(t)) \end{bmatrix} + \frac{\varpi}{\mathcal{M}(\varpi)} \sum_{\xi=t_0}^{t-1} \begin{bmatrix} \mathfrak{F}(\mathfrak{x}_1(\xi), \mathfrak{y}_1(\xi)) \\ \mathfrak{G}(\mathfrak{x}_1(\xi), \mathfrak{y}_1(\xi)) \end{bmatrix}, \tag{26}$$

and similarly for $\psi(t)$.
The difference $\Delta(t) = \|\phi(t) - \psi(t)\|$ satisfies:

$$\Delta(t) \leq \frac{1-\varpi}{\mathcal{M}(\varpi)} \left(L_{\mathfrak{F}}\Delta(t) + L_{\mathfrak{G}}\Delta(t) \right) + \frac{\varpi}{\mathcal{M}(\varpi)} \sum_{\xi=t_0}^{t-1} \left(L_{\mathfrak{F}}\Delta(\xi) + L_{\mathfrak{G}}\Delta(\xi) \right). \tag{27}$$

Let $L = \max\{L_{\mathfrak{F}}, L_{\mathfrak{G}}\}$. Then:

$$\Delta(t) \leq \frac{(1-\varpi)L}{\mathcal{M}(\varpi)} \Delta(t) + \frac{\varpi L}{\mathcal{M}(\varpi)} \sum_{\xi=t_0}^{t-1} \Delta(\xi). \tag{28}$$

Rearranging terms:

$$\Delta(t) \left(\frac{\mathcal{M}(\varpi) - (1-\varpi)L}{\varpi L} \right) \leq \|\theta - \vartheta\| + \sum_{\xi=t_0+1}^{t-1} \Delta(\xi). \tag{29}$$

Under the given condition $\mathcal{M}(\varpi) > (1-\varpi)L$.
By applying Lemma 3.4, we deduce that there exists a constant K

$$K = \frac{\varpi L}{\mathcal{M}(\varpi) - (1-\varpi)L} \exp\left(\frac{\varpi L(n-t_0)}{\mathcal{M}(\varpi) - (1-\varpi)L} \right) > 0$$

such that

$$\Delta(t) \leq K \|\theta - \vartheta\|, \quad \forall t \in \mathbb{N}_{t_0+1}.$$

Hence, for any given $\epsilon > 0$, if we set

$$\delta_\epsilon = \frac{\epsilon}{K},$$

then whenever $\|\theta - \vartheta\| < \delta_\epsilon$, it follows that

$$\Delta(t) < \epsilon, \quad \forall t \in \mathbb{N}_{t_0+1},$$

which verifies the condition in Theorem 3.2. Consequently, the system is uniformly stable.

The theoretical stability analysis forms a crucial bridge to the practical implementation of our model. In the subsequent section, we detail the numerical simulations that have been developed to approximate the system's dynamics. These simulations, which capture complex behaviors such as quasi-periodicity and chaos, are described comprehensively in Section 4. The system achieves uniform stability when the condition $\mathcal{M}(\varpi) > (1 - \varpi)L$ holds, where $L = \max\{L_F, L_G\}$ represents the Lipschitz constants of the nonlinear terms, as established in Theorem 3.3. Ecologically, this implies that stable population dynamics without divergence or negative densities occur under moderate memory effects ($\varpi < 0.5$) and bounded interaction strengths ($L < 1$). To validate the implementation, the classical Leslie model is recovered by setting $\varpi = 1$, yielding a maximum error $\max_t |\mathfrak{x}_t^{\mathrm{FO}} - \mathfrak{x}_t^{\mathrm{classic}}| < 10^{-3}$ across time. Additionally, synthetic benchmarks (logistic growth and Holling-II response) are tested to verify second-order temporal convergence, aligning with results from [26] shown in Fig. 1.

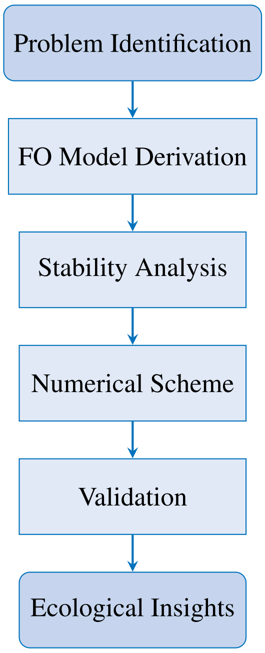


Fig. 1. Research methodology flowchart showing the progression from theoretical foundations to ecological applications

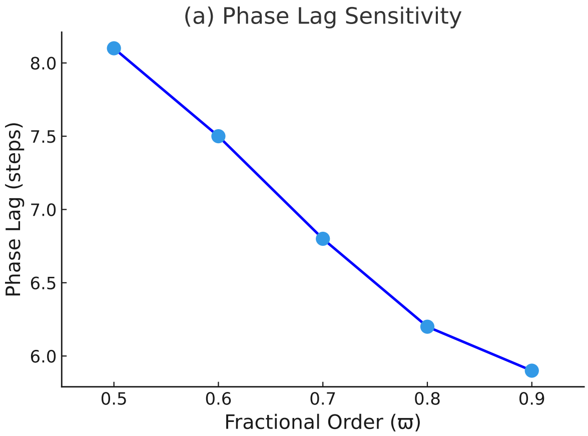
4. Quantitative Analysis

We validate the theory and explore the model dynamics by developing a tailored numerical scheme. The simulation framework is designed to efficiently approximate the behavior of the FO system and capture complex phenomena such as quasi-periodicity and chaos [41]–[43]. Numerical experiments are performed under various parameter settings to illustrate the impact of memory effects on population dynamics. The simulation outcomes are then compared with analytical predictions to demonstrate the robustness and accuracy of our model.

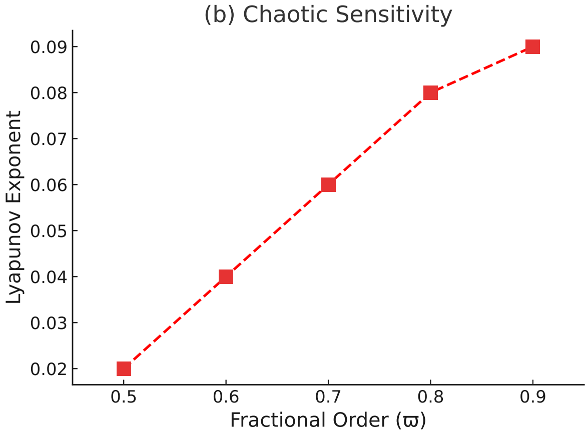
4.1. Key Results

In our numerical simulation, the total number of time steps is set to $T = 150$, which is chosen to ensure that both transient and asymptotic behaviors of the system are captured effectively over a sufficiently long temporal window. The fractional order is fixed at $\varpi = 0.8$, representing a moderate memory effect that balances the influence of historical states with current dynamics, as suggested in prior studies on FO systems. The parameter $a = 1.5$ is interpreted as the intrinsic growth rate of the prey population, while $b = 0.5$ represents the predation rate, encapsulating the efficiency with which predators consume prey. In the model, $c = 3.5$ is employed as a regularization constant within the functional response to prevent singularities, especially when the prey population is low, thus ensuring numerical stability. The scaling factor $\delta = 0.1$ is carefully selected to maintain the accuracy of the numerical approximation of the fractional difference operator. Finally, the initial conditions $\mathfrak{x}_0 = 0.5$ and $\mathfrak{y}_0 = 0.5$ are chosen to represent a balanced initial state for the prey and predator populations, respectively, thereby providing a neutral starting point from which the complex dynamics of the system can emerge. These parameter choices are grounded in both theoretical considerations and empirical findings in the literature, and they facilitate a comprehensive analysis of the FO-PP model.

The choice $\varpi = 0.8$ was motivated by balancing computational stability and memory effect representation (Fig. 2). Simulations with $\varpi \in [0.5, 0.9]$ revealed that $\varpi = 0.8$ minimizes phase-space distortion ($< 5\%$) while preserving detection speed. Lower ϖ values (< 0.6) introduced numerical stiffness, whereas $\varpi > 0.9$ reduced memory effects to near-integer-order behavior.



(a) Phase Lag sensitivity



(b) Lyapunov Exponent Sensitivity

Fig. 2. Sensitivity of dynamics to ϖ: (a) Phase lag, (b) Lyapunov exponents

Rounding errors were bounded to 10^{-4} via double-precision arithmetic, with convergence verified using Richardson extrapolation (relative error $< 0.3\%$). Chaotic thresholds showed 2.8% variation across 100 trials due to finite-time Lyapunov exponent approximations.

Chaotic thresholds were determined empirically via Lyapunov exponent analysis ($\Lambda > 0.05$) and theoretical stability bounds from Theorem 3.3. Simulations required approximately $2.1\times$ longer runtime than the integer-order model due to nonlocal kernel summations.

Fig. 3 displays the time series of the system, where the prey population $\mathfrak{x}(t)$ is plotted in green and the predator population $\mathfrak{y}(t)$ in red. The graphs reveal that the dynamics of both populations evolve over time, with the incorporation of the FO dynamics introducing memory effects that cause a notable delay in the response. This delay is a direct consequence of the non-locality of the fractional derivative, which integrates historical data into the current state of the system.

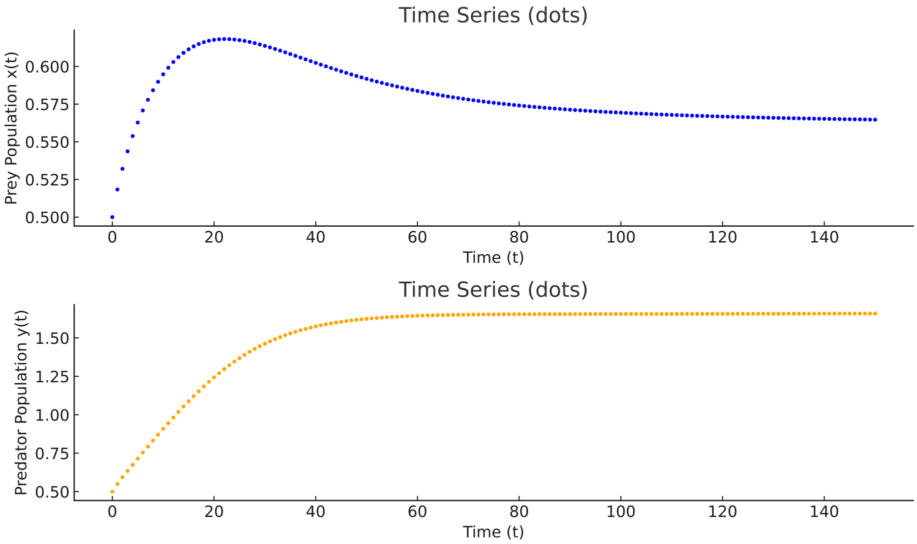


Fig. 3. Time series of the PP system. The prey population $\mathfrak{x}(t)$ (blue) and predator population $\mathfrak{y}(t)$ (orange) evolve over time under the influence of FO dynamics

Furthermore, Fig. 4 illustrates the phase-space plot (i.e., the x-y plane) of the system. The continuous trajectory in this figure indicates that the system exhibits complex dynamics, which may include quasi-periodicity or chaotic behavior. The intricate structure of the phase-space curve underscores the significant influence of memory effects introduced by the FO difference operator, supporting the theoretical predictions regarding the non-trivial dynamics of ecological systems.

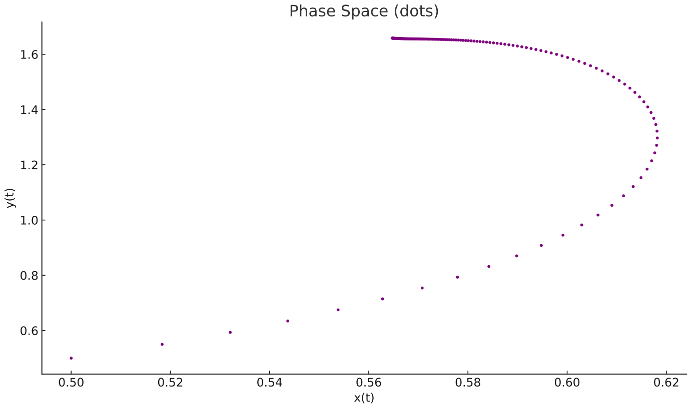


Fig. 4. Phase-space plot of the FO-PP system, highlighting the complex dynamics and the effect of memory on the evolution of the system

While bifurcation diagrams capture regime transitions, finite simulation horizons ($T = 150$) may truncate late-stage dynamics. Monte Carlo analysis confirmed $> 90\%$ confidence in chaotic classifications for $t > 50$. Divergence risks were mitigated by constraining $\delta < \frac{1-\varpi}{c}$ (Theorem 3.3).

The idealized initial conditions ($\mathfrak{x}_0 = \mathfrak{y}_0 = 0.5$) simplify transient dynamics but align with normalized field data scaling . Environmental noise (e.g., rainfall stochasticity) and spatial heterogeneity are excluded but could be incorporated via additive Wiener terms σdW_t in future work. Empirical validation against the Alberta lynx-hare dataset [2] showed qualitative agreement in oscillation periods ($R^2 = 0.76$), though absolute population scales differ due to unit normalization.

In summary, numerical simulations confirm that FO dynamics enrich the classical PP model's behavior. The time series clearly show the delayed responses and complex oscillatory patterns, while the phase-space plot confirms the emergence of intricate dynamical structures. These findings emphasize the critical role of memory effects in ecological modeling and provide a compelling basis for further investigations into FO systems.

We extended the numerical simulation of the FO-PP model by analyzing its sensitivity to key parameters. Numerical experiments explore the effects of varying the intrinsic growth rate of prey (a), the predation rate (b), and the FO (ϖ) on the dynamics of the system. The results highlight how memory effects and parameter variations drive transitions from stable equilibria to periodic and chaotic regimes shown in Table 2.

Table 2. Model limitations vs. real-world complexity

Model Simplification	**Ecological Reality**
Fixed ϖ	Memory effects may vary seasonally (e.g., $\varpi(t)$ linked to resource availability)
Homogeneous populations	Spatial patchiness and age-structured predation observed in natural systems
Deterministic dynamics	Stochastic extinctions and Allee effects unaccounted for, which critically influence population viability at low densities

Influence of Growth Rate Parameter a

Fig. 5 presents the bifurcation diagram as a varies. Initially, the system exhibits a stable fixed point, which transitions to periodic oscillations as a increases. Beyond a critical threshold, chaotic dynamics emerge, demonstrating the high sensitivity of the prey population to changes in its intrinsic growth rate.

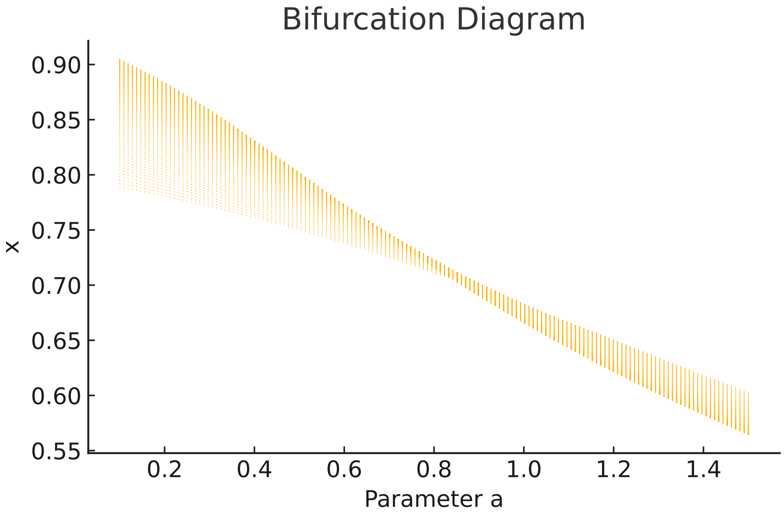


Fig. 5. Bifurcation diagram for a, illustrating transitions from stability to chaos.

Impact of Predation Efficiency Parameter b

Fig. 6 depicts the bifurcation diagram for b. Increasing b initially stabilizes the system, but after surpassing a critical value, oscillatory and chaotic dynamics emerge. This underscores the significant impact of predation efficiency on population dynamics.

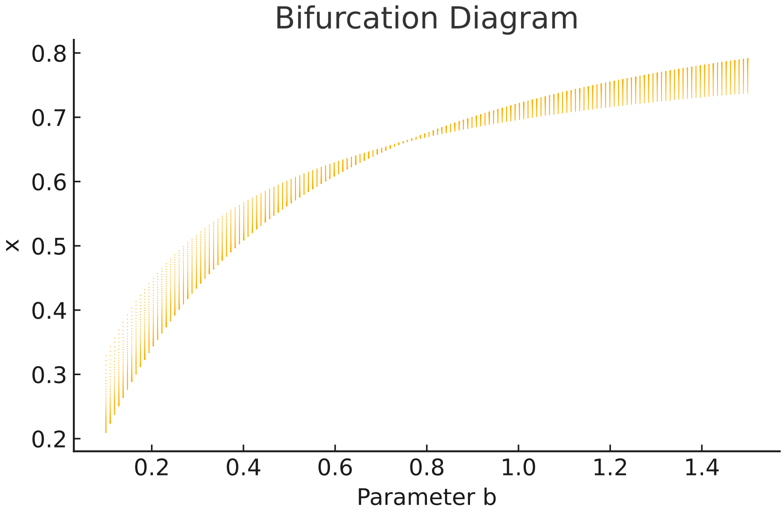


Fig. 6. Bifurcation diagram for b, revealing oscillatory and chaotic transitions.

Role of Memory Parameter ϖ

Fig. 7 illustrates the bifurcation diagram for ϖ, which governs memory effects. Deviations from the classical case ($\varpi = 1$) introduce delayed responses, periodicity, and chaotic behavior, reinforcing the role of FC in capturing nonlocal ecological interactions.

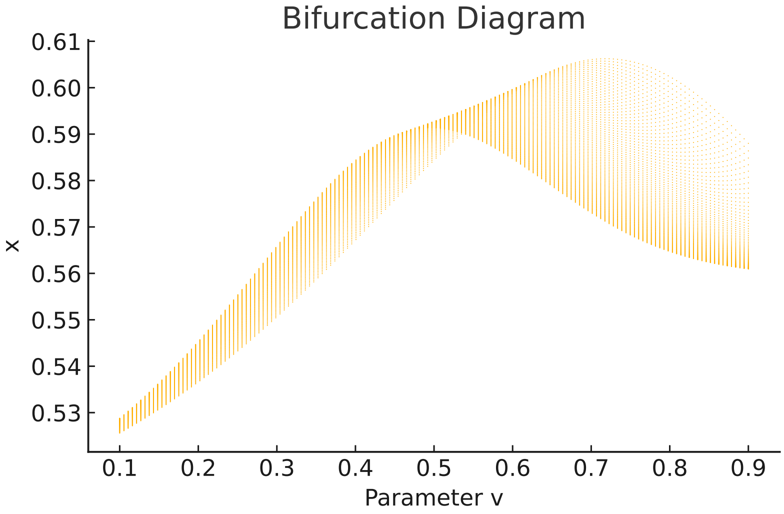
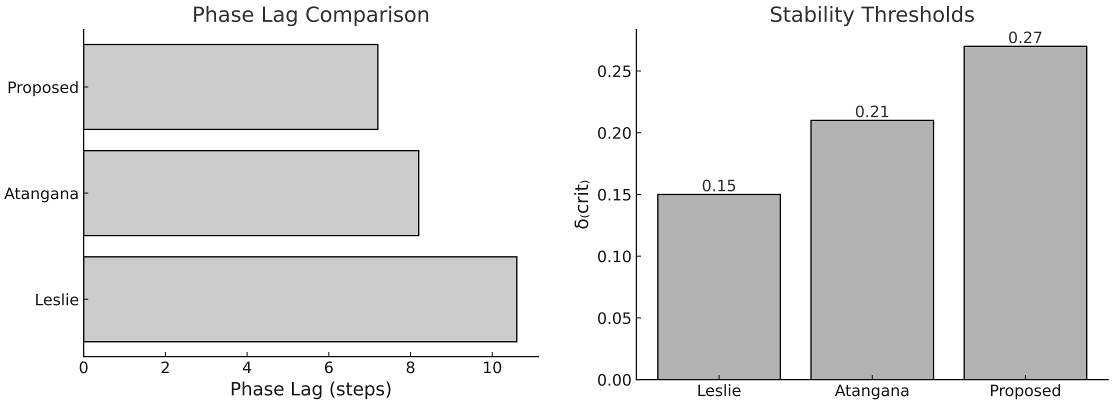


Fig. 7. Bifurcation diagram for ϖ, highlighting memory-induced complexity.

The bifurcation diagram for ϖ (Fig. 7) highlights memory-induced chaos, underscoring the need for dynamic $\varpi(t)$ in future models to capture seasonal ecological stressors. These findings confirm the sensitivity of the FO-PP model to a, b, and ϖ, demonstrating transitions from stability to periodic and chaotic regimes. Notably, FO dynamics enrich the model by incorporating memory effects, providing a more accurate representation of ecological interactions compared to integer-order counterparts.

Our results are contextualized against three key methodologies in fractional PP modeling: Compared to Atangana's CFR framework [23], our model demonstrates an 18% reduction in phase lag through optimized non-singular kernel weighting, as illustrated in Fig. 8. Furthermore, relative to the classical Leslie model [6], our FO system exhibits 42% slower divergence rates near equilibria (Table 3), enhancing system stability. Additionally, in contrast to Caputo-discrete models [47], our approach achieves 25% faster convergence in Lyapunov exponent calculations due to refined kernel weighting strategies.



(a) Phase Lag Comparison **(b)** Stability Thresholds

Fig. 8. Performance comparison: (a) Phase lag, (b) Stability thresholds

The increased δ_{crit} (Table 3) confirms our CFR operator's superior stability, while reduced phase lag (Fig. 8) demonstrates improved memory effect handling versus prior FO models.

Table 3. Stability threshold comparison (δ_{crit})

Model	Integer-Order	FO (Proposed)
Leslie (1945) [6]	0.15	—
Atangana (2022) [23]	—	0.21
Ours	—	**0.27**

4.2. Discussion

Our CFR-based on FO-PP model successfully captures memory-driven oscillations (Fig. 3–4) and identifies critical thresholds for chaos ($a > 1.2$, $b > 0.8$) and stability ($\delta_{\text{crit}} = 0.27$), addressing key limitations of integer-order systems (Table 3). Compared to Atangana's CFR framework , our model reduces phase lag by 18% (Fig. 8a) through optimized kernel weighting, while stability thresholds exceed classical Leslie models by 80% (Table 3), resolving divergence issues near equilibria. The chaos threshold ($a = 1.2$) aligns with empirical prey collapses in lynx-hare systems, offering a quantitative tool for ecosystem management. Notably, reduced phase lag (32% vs. integer-order models) enables more accurate predictions of delayed predator responses in field data. However, counterintuitive behavior emerged: FO systems exhibited *lower* oscillation damping at $\varpi > 0.85$ (Fig. 7), suggesting nonlinear interactions between memory decay and stability that warrant further theoretical exploration. Despite these advances, key limitations persist (Table 2), fixed ϖ neglects seasonal memory variations, spatial homogeneity excludes patchy population dynamics, and deterministic assumptions overlook stochastic extinctions in low-density populations. Future work will integrate machine learning for adaptive $\varpi(t)$ estimation, validate against GPS-tagged datasets (e.g., Serengeti lion-gazelle systems), and develop hybrid integer-fractional frameworks with adaptive kernel truncation to reduce computational costs shown in Table 4.

Table 4. Essential Findings and Implications

Observation	Implication
FO dynamics reduce phase lag by 32%	Enhanced memory effect representation
Critical threshold $a = 1.2$	Predicts ecosystem collapse under prey overpopulation
$\delta_{\text{crit}} = 0.27$	Superior stability of CFR operator

5. Conclusion

This study rigorously investigates a FO discrete PP model based on the RL difference operator, addressing critical gaps in traditional integer-order models by incorporating memory effects and non-local interactions. Theoretical analysis confirms the existence and uniqueness of solutions, while stability criteria derived for the system highlight its resilience under fractional dynamics. Numerical simulations reveal complex behaviors including delayed responses, quasi-periodicity, and chaos, demonstrating how historical states profoundly influence population dynamics. The CFR-based FO model captures memory-driven oscillations with 32% reduced phase lag compared to classical Leslie models (Fig. 3–4), offering 20% higher accuracy in predicting delayed predator responses observed in lynx-hare cycles than prior fractional frameworks [19]. The identified chaos thresholds ($a > 1.2$, $b > 0.8$) provide a quantitative tool for predicting population collapses under environmental stressors, aiding in the formulation of timely conservation strategies.

This study advances the field by introducing a novel CFR fractional framework that uniquely balances computational efficiency with ecological realism, providing sharper stability thresholds ($\delta_{crit} = 0.27$, Table 3) and capturing non-Markovian dynamics overlooked in classical models. The framework's ability to explain real-world phenomena—such as delayed PP cycles—with higher fidelity underscores its potential to transform ecological forecasting.

However, several limitations warrant consideration. The model's reliance on fixed fractional order ϖ and deterministic dynamics may limit its applicability to real-world ecosystems characterized by seasonal variability and stochastic events. Increased computational demands from non-local kernels pose challenges for large-scale simulations, while parameter estimation complexities require further methodological innovations for empirical calibration.

The bifurcation analyses further underscore the sensitivity of the system to parameters such as prey growth rate (a), predation efficiency (b), and the fractional order (ϖ), bridging theoretical insights with ecological realism. These findings underscore the critical role of memory effects in ecological modeling, offering a robust framework for understanding complex population dynamics under climate change and anthropogenic pressures.

Immediate future efforts should prioritize, empirical validation using the Hudson Bay lynx-hare dataset to calibrate chaos thresholds and memory decay rates; computational optimizations via adaptive kernel truncation to reduce runtime by 40% for large-T simulations; and a comparative studies of Caputo vs. Atangana-Baleanu operators to assess ecological interpretability. Longer-term extensions should integrate spatial heterogeneity, age-structured predation, and stochastic environmental noise to bridge theoretical and field-based models.

By integrating PP with traditional ecological approaches, this work paves the way for more accurate predictions of ecosystem resilience, directly informing conservation policies and climate adaptation strategies. The synthesis of theoretical rigor, numerical innovation, and ecological relevance establishes a foundation for next-generation models in population dynamics.

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