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Stability Analysis of a Fractional-Order Lengyel–Epstein Chemical Reaction Model

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ABSTRACT

In this paper, we stady a mathematical model based on a system of fractional-order differential equations to describe the dynamics of the Lengyel–Epstein chemical reaction, which is well known for exhibiting oscillatory behavior. The use of fractional derivatives allows in chemical processes compared to classical integer-order models. We specifically focus on analyzing the stability of the system's positive equilibrium point by applying fractional calculus techniques. The stability conditions are derived and discussed in the context of the fractional-order parameters. To validate the theoretical findings, we perform numerical simulations using the Forward Euler method adapted for fractional-order systems. These simulations illustrate the impact of the fractional order on the system's dynamic behavior and confirm the analytical results regarding equilibrium stability.

1. Introduction

Recent advancements in numerical methods for fractional differential equations have yielded a variety of effective techniques and models. Farraj et al. [1] and Anakira et al. [2] developed optimized and algorithmic approaches for solving conformable and Volterra integro-differential equations, respectively. Berir [3] applied a novel method to study stochastic effects in fractional systems, while Batiha et al. proposed numerical schemes such as the trapezoidal method for fractional initial value problems [4] and computational methods for neutron diffusion in nuclear reactors [5]. Other studies have introduced Laplace–Caputo-based RKDM techniques for nonlinear problems [6], finite difference methods for time–space fractional models [7], and multi-group neutron diffusion systems [8]. Additionally, the stochastic behavior of population dynamics has been modeled using fractional formulas [9]. Bouchenak et al. [10], [11] generalized nonlinear fractional models, including Bernoulli and Cauchy–Euler equations under modified conformable frameworks. Finally, Lamamri et al. [12] explored the application of Caputo and conformable derivatives in analyzing nonlinear beam deflection problems, further highlighting the versatility of fractional calculus in mathematical modeling,

see [13], [14] to get more details.

Oscillating chemical reactions such as the Belousov–Zhabotinsky and Briggs–Rauscher reactions are exceptional due to their nonlinear dynamic behavior, making them classical examples of non-equilibrium thermodynamic systems. These reactions have been the subject of extensive mathematical modeling aimed at understanding their underlying mechanisms. However, the resulting models are often complex and analytically challenging due to the multitude of interacting species and nonlinear rate laws involved [15]–[17]. In contrast, the Lengyel–Epstein reaction, which involves iodine (I^-), malonic acid (MA), and chlorine dioxide (CLO_2^-), provides a more simplified yet still powerful framework for studying oscillatory chemical behavior. Derived from the chlorite–iodide–malonic acid (CIMA) reaction, this model captures essential features of pattern formation and temporal oscillations, and has been widely used as a prototype for reaction–diffusion systems [18]. The CIMA reaction can be described by three chemical reaction schemes as follows

$$\begin{cases} I_2 + MA \to IMA + H^+ + I^- \\ I^- + CLO_2 \to \frac{1}{2} I_2 + CLO_2^- \\ 4I^- + 4H^+ + CLO_2^- \to 2I_2 + Cl^- + 2H_2O. \end{cases} \tag{1}$$

By applying empirical rate laws and omitting constant coefficients, the reaction kinetics of the chlorite–iodide–malonic acid (CIMA) system can be simplified into the conventional Lengyel–Epstein model. This reduced model involves two dependent variables, U and V, which represent the time evolution of the concentrations of I^- and CLO_2^-, respectively. The Lengyel–Epstein model has been the subject of extensive mathematical investigation due to its ability to capture essential features of nonlinear chemical dynamics, including oscillations and pattern formation. Several studies have established sufficient conditions for both local and global asymptotic stability of its equilibrium points [19]–[25]. Furthermore, diffusion-driven instability, commonly referred to as Turing instability, has been rigorously analyzed in works such as [26]–[28], which provide criteria under which spatial patterning emerges. The model's capacity to exhibit Hopf bifurcations—indicating transitions to temporal oscillations—has also been discussed in detail in [29]–[34]. In addition, a wide range of modified versions of the original system have been explored in the literature [35]–[39], with the goal of relaxing classical assumptions or extending the model's applicability to more complex chemical and biological phenomena. the proposed rate equations are given by

$$\begin{cases} \frac{dU}{dt} = M - NU - 4P\left(\frac{UV}{\alpha + U^2}\right) \\ \frac{dV}{dt} = PU - P\left(\frac{UV}{\alpha + U^2}\right), \end{cases} \tag{2}$$

Where $M, N, P > 0$. After all these operations, the Lengyel–Epstein model is as the following [40]:

$$\begin{cases} \frac{du}{dt} = l - u - \frac{4uv}{1+u^2} \\ \frac{dv}{dt} = mu\left(1 - \frac{v}{1+u^2}\right). \end{cases} \tag{3}$$

Model (3) describes an integer-order system that incorporates a first-order derivative with respect to the time variable, t. This derivative captures the immediate rate of change in the reactions. However, biochemical processes are inherently complex and often influenced not only by their current state but also by their past dynamics. To better capture these memory-dependent behaviors, fractional-order differential equations provide a more appropriate analytical framework [41]–[44].

This paper is systematized into four sections. The introduction is the first section in which we intricate some work of the Lengyel–Epstein model in kinetic studies. In Section 2, we will elaborate notations related to the concept of FDEs. In Section 3, chemical reaction model of Lengyel–Epstein incorporating fractional-order dynamics. Numerical imitations are offered to validate the main outcomes and conclusion is drawn in Section 4.

2. Fractional Calculus

Let us review some fundamental definitions [45], [46] related to the Caputo differential operator in fractional calculus.

Definition 2.1 [47] Suppose that $\alpha > 0, t > a, \alpha, a, t \in \mathbb{R}$. The Caputo fractional derivative is given by

$$\begin{aligned} {}_a^C D_t^\alpha f(t) &= \frac{1}{\Gamma(n-\alpha)} \int_a^\alpha \frac{f^n(\xi)}{(t-\xi)^{\alpha-1-n}} d\xi, \\ n-1 &< \alpha < n \in \mathbb{N} \quad D = \frac{d}{dt}, \end{aligned} \tag{4}$$

Where Γ representing the gamma function.

Theorem 2.2 Note that the constant (s^*, u^*) is an equilibrium point for the Caputo fractional non-autonomous dynamic system

$$\begin{cases} {}_{t_0}^C D_t^\alpha s(t) = F_1(s,u), & \text{in } \mathbb{R}_+ \\ {}_{t_0}^C D_t^\beta u(t) = F_2(s,u), & \text{in } \mathbb{R}_+, \end{cases} \tag{5}$$

if and only if

$$F_1(s^*, u^*) = F_2(s^*, u^*) = 0 \tag{6}$$

Lemma 2.3 [48] The asymptotic stability of the point (s^*, u^*) established is subject to

$$|\arg(\lambda_1)| > \frac{\alpha\pi}{2} \text{ and } |\arg(\lambda_2)| > \frac{\beta\pi}{2}, \tag{7}$$

Where $\alpha, \beta \in (0, 1]$ and $\arg(.)$ is the argument of a complex number, $\lambda_i \ (i = 1, 2)$ are the eigenvalues of the Jacobian matrix $J(s^*, u^*)$.

2.1. Local Stability from an ODE Perspective

Consider the following simple linear two-component ODE system (see [49]):

$$Q_t = AQ, \tag{8}$$

Where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad Q_1 = \begin{pmatrix} x \\ y \end{pmatrix} \tag{9}$$

It is well known that the asymptotic behavior is heavily dependent on the eigenvalues of A denoted by λ_1 and λ_2 and A being nonsingular and that is $(x, y) = (0, 0)$. The qualitative properties of the solutions to system (8) is the asymptotic behavior of the solutions as $t \to +\infty$. A summary of this dependency is given in Table 1. The first stability case, which is the asymptotically stable node, can be guaranteed if:

$$\text{tr}(A) = a_{11} + a_{22} < 0 \quad \text{and} \quad \det(A) = a_{11}a_{22} - a_{12}a_{21} > 0$$

Table 1. The asymptotic behavior of solutions to system (8)

Eigenvalues	Type of equilibrium
$\lambda_i \in \mathbb{R}, \lambda_i < 0, i = 1, 2$	Asymptotically stable node
$\lambda_i \in \mathbb{R}, \lambda_i > 0, i = 1, 2$	Unstable node
$\lambda_i \in \mathbb{R}, \lambda_i < 0, i = 1, 2$	Unstable saddle
$\lambda = \alpha \pm i\beta, \alpha < 0$	Asymptotically stable node
$\lambda = \alpha \pm i\beta, \alpha > 0$	Unstable focus
$\lambda = \pm i\beta$	Stable center

Theorem 2.4 (**Routh-Hurwitz Criteria**) Given the characteristic polynomial

$$G(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + ... + a_{n-1}\lambda + a_n,$$

Where the coefficients a_i are real constants, $i = 1, ..., n$, the $n-$ Hurwitz matrices by the coefficients a_i of the upper polynomial are

$$H_1 = (a_1), \; H_2 = \begin{pmatrix} a_1 & 1 \\ a_3 & a_2 \end{pmatrix}, \; H_3 = \begin{pmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{pmatrix},$$

$$H_n = \begin{pmatrix} a_1 & 1 & 0 & 0 & . & 0 \\ a_3 & a_2 & a_1 & 1 & . & 0 \\ a_5 & a_4 & a_3 & a_1 & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & a_n \end{pmatrix},$$

Where $a_j = 0$ if $j > n$. The roots of polynomial $G(\lambda)$ are negative or have negative real parts, iff the determinants of all Hurwitz matrices are positive: $\det H_j > 0$, $j = 1, 2, ..., n$. the Routh-Hurwitz Criteria simplify to

$$\begin{aligned} n = 2: \; & a_1 > 0 \text{ and } a_2 > 0. \\ n = 3: \; & a_1 > 0, \; a_3 > 0 \text{ and } a_1a_2 > a_3. \\ n = 4: \; & a_1 > 0, \; a_3 > 0, \; a_4 > 0 \text{ and } a_1a_2a_3 > a_3^2 + a_1^2a_4. \\ n = 5: \; & a_i > 0, \; i = 1, 2, 3, 4, 5 > 0, \; a_1a_2a_3 > a_3^2 + a_1^2a_4 \text{ and} \\ & (a_1a_4 - a_5)\left(a_1a_2a_3 - a_3^2 - a_1^2a_4\right) > a_5\left(a_1a_2 - a_3\right)^2 + a_1a_5^2. \end{aligned}$$

Theorem 2.5 [50] The equilibrium point of the fractional differential equation's system is asymptotically stable if all the eigenvalues obtained from the polynomial

$$\det\left(\text{diag}\left(\lambda^{\omega\alpha_1}, \lambda^{\omega\alpha_2}, ..., \lambda^{\omega\alpha_n}\right) - J(E)\right) = 0 \text{ satisfy } \left|\arg(\lambda)\right| > \frac{\gamma\pi}{2}, \tag{10}$$

Where $J(E)$ is Jacobian matrix evaluated at equilibrium point E.

3. Fractional-Order Dynamics in the Lengyel–Epstein Reaction Model

The model proposed in this study is the multi-order fractional order system of differential equations model of the Lengyel-Epstein model proposed in [51] without diffusion. As follows

$$\begin{cases} \frac{d^\alpha s}{dt^\alpha} = \Lambda - s - \frac{4su}{1+e^s} = F_1(s, u) & \text{in } \mathbb{R}_+ \times \Omega. \\ \frac{d^\beta u}{dt^\beta} = ms\left(1 - \frac{u}{1+e^s}\right) = F_2(s, u) & \text{in } \mathbb{R}_+ \times \Omega. \end{cases} \tag{11}$$

Where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$ and Λ and m are strictly positive constants, $\alpha,\ \beta \in (0,1]$ is the fractional order with Caputo fractional derivative over $(0,\infty)$. We assume the nonnegative initial conditions shown in Fig. 1

$$0 \le s(0,x) = s_0(x), 0 \le u(0,x) = u_0(x) \qquad \text{in } \Omega,$$

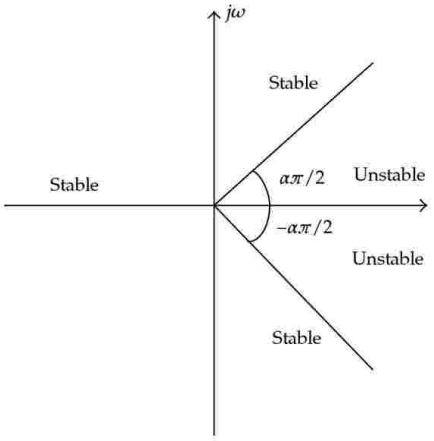

Fig. 1. Stability region of fractional order system (11).

and impose homogeneous Neumann boundary conditions

$$\frac{\partial s}{\partial \upsilon} = \frac{\partial u}{\partial \upsilon} = 0 \qquad \text{on } \mathbb{R}_+ \times \Omega,$$

Where ν is the unit outer normal to $\partial\Omega$.

Theorem 3.1 The region **V** is a positively invariant for the system (11)

$$\mathbf{V} = \left\{(s,u) \in \mathbb{R}^2_+ / \ s \ge 0,\ u \ge 0\right\},$$

with initial conditions $s(0) > 0$ and $u(0) > 0$.

Proposition 3.2 System (11) has a unique equilibrium $(s^*, u^*) = (\kappa, 1 + e^{\kappa})$ with $\kappa = \frac{\Lambda}{5}$.

Proof 1 To find the equilibrium point of (11), we put

$$\frac{d^\alpha s}{dt^\alpha} = \frac{d^\beta u}{dt^\beta} = 0.$$

So we have

$$\begin{cases} \Lambda - s - \frac{4su}{1+e^s} = 0 \\ ms\left(1 - \frac{u}{1+e^s}\right) = 0. \end{cases} \tag{12}$$

From the second equation of (12), we have either $s = 0$ or $1 - \frac{u}{1+e^s} = 0$. If $s = 0$, then the first equation of (12) gives $\Lambda = 0$, which is false because $\Lambda > 0$.

So $s \neq 0$, we have $u = 1 + e^s$. Then the equilibrium point is $(s^*, u^*) = (\kappa, 1 + e^{\kappa})$ with $\kappa = \frac{\Lambda}{5}$. This completes the proof.

Proposition 3.3 For the fractional-order system (11),

• Subject to $\Delta_\lambda = \left(\frac{4\kappa e^\kappa - m\kappa - 5(1+e^\kappa)}{1+e^\kappa}\right)^2 - \frac{20m\kappa}{1+e^\kappa} \geq 0$.

The equilibrium point (s^*, u^*) is asymptotically stable if $\mathrm{tr}J(E) < 0$ and unstable if $\mathrm{tr}J(E) > 0$, where

$$J(E) = \begin{pmatrix} -\frac{5(1+e^\kappa)-4\kappa e^\kappa}{1+e^\kappa} & -\frac{4\kappa}{1+e^\kappa} \\ \frac{m\kappa e^\kappa}{1+e^\kappa} & -\frac{m\kappa}{1+e^\kappa} \end{pmatrix}.$$

• If $\Delta_\lambda < 0$, then (s^*, u^*) is asymptotically stable if $\mathrm{tr}J(E) < 0$.

Proof 2 The functions of the system (11) can be determined as below:

$$\begin{cases} F_1(s,u) = \Lambda - s - \frac{4su}{1+e^s} \\ F_2(s,u) = ms\left(1 - \frac{u}{1+e^s}\right). \end{cases} \tag{13}$$

The Jacobian of (13) is

$$\begin{aligned} J(s,u) &= \begin{pmatrix} F_{1_s} & F_{1_u} \\ F_{2_s} & F_{2_u} \end{pmatrix} \\ &= \begin{pmatrix} -1 - \frac{4u(1+e^s-se^s)}{(1+e^s)^2} & -\frac{4s}{1+e^s} \\ m - \frac{mu(1+e^s-se^s)}{(1+e^s)^2} & -\frac{ms}{1+e^s} \end{pmatrix}. \end{aligned} \tag{14}$$

Then the Jacobian matrix (14) at $E = (s^*, u^*)$, we have

$$J(E) = \begin{pmatrix} -\frac{5(1+e^\kappa)-4\kappa e^\kappa}{1+e^\kappa} & -\frac{4\kappa}{1+e^\kappa} \\ \frac{m\kappa e^\kappa}{1+e^\kappa} & -\frac{m\kappa}{1+e^\kappa} \end{pmatrix}. \tag{15}$$

Its determinant and trace are given by

$$\det J(E) = \frac{5m\kappa}{1+e^\kappa}$$

and

$$\mathrm{tr}J(E) = \frac{4\kappa e^\kappa - m\kappa - 5(1+e^\kappa)}{1+e^\kappa},$$

respectively. The characteristic equation of the Jacobian matrix $J(E)$ is

$$\lambda^2 - \mathrm{tr}J(E)\lambda + \det J(E) = 0,$$

The discriminant is

$$\Delta_\lambda = (\mathrm{tr}J(E))^2 - 4\det J(E).$$

We study the different cases separately. Referring to [52]

• First if $\Delta_\lambda > 0$, then the eigenvalues λ_1 and λ_2 are

Real where $\lambda_1 = \frac{1}{2}\left[\mathrm{tr}J(E) + \sqrt{\Delta_\lambda}\right]$ and $\lambda_2 = \frac{1}{2}\left[\mathrm{tr}J(E) - \sqrt{\Delta_\lambda}\right]$.
Note that $\det J(E) > 0$. Hence the negativity of the eigenvalues rests on the sign of the trace.

(i) If $\mathrm{tr}J(E) < 0$, then $\mathrm{tr}J(E) - \sqrt{\Delta_\lambda} < 0$, leading to

$$\lambda_2 = \frac{1}{2}\left[\mathrm{tr}J(E) - \sqrt{\Delta_\lambda}\right] < 0$$

1. and therefore, $\arg(\lambda_2) = \pi$. Since both eigenvalues are real, $\mathrm{tr}J(E) < 0$ and $\det J(E) > 0$, it is clear that $\arg(\lambda_1) = \pi > \frac{\alpha\pi}{2}$ and $\arg(\lambda_2) = \pi > \frac{\beta\pi}{2}$ with $\alpha, \beta \in]0, 1]$. It follows that $E(s^*, u^*)$ is asymptotically stable.

(ii) If $\mathrm{tr}J(E) > 0$, then $\mathrm{tr}J(E) - \sqrt{\Delta_\lambda} > 0$, leading to

$$\lambda_2 = \frac{1}{2}\left[\mathrm{tr}J(E) - \sqrt{\Delta_\lambda}\right] > 0.$$

Note that $\lambda_1 > 0$, then $\arg(\lambda_1) = \arg(\lambda_2) = 0$. So $E(s^*, u^*)$ is asymptotically stable.

(iii) If $\mathrm{tr}J(E) = 0$, then $\Delta_\lambda > 0$, leading to $-4\det J(E) < 0$ which is a contradiction. Hence this case does not show up.

- The seconde case of the discriminant $\Delta_\lambda = 0$. Since $\det J(E) > 0$, then it is impossible that $\mathrm{tr}J(E) = 0$. The eigenvalues reduce to

$$\lambda_{1,2} = \frac{1}{2}\mathrm{tr}J(E).$$

The sign of the eigenvalues is identical to that of the trace. Consequently $E(s^*, u^*)$ is asymptotically stable for all $\alpha, \beta \in]0, 1]$, if $\mathrm{tr}J(E) < 0$ and unstable if $\mathrm{tr}J(E) > 0$.

- Finally if the discriminant $\Delta_\lambda < 0$, then$\lambda_{1,2} = \frac{1}{2}\left[\mathrm{tr}J(E) \pm i\sqrt{-\Delta_\lambda}\right]$.

We have three cases:

- If $\mathrm{tr}J(E) < 0$, then the system is asymptotically stable at the equilibrium point.
- If $\mathrm{tr}J(E) = 0$, then $\left|\arg\left(\frac{1}{2}\left[\pm i\sqrt{-\Delta_\lambda}\right]\right)\right| = \frac{\pi}{2}$, Hence for $\alpha < 1$, $\beta < 1$, is asymptotically stable at the equilibrium point. In the special case, if $\alpha = \beta = 1$ then the system is unstable at the equilibrium point.
- If $\mathrm{tr}J(E) > 0$, then the system is unstable at the equilibrium point. The proof is complete.

Now we are going to use the eigenfunction expansion method. From (15) we have From the equation

$$\det\left[\mathrm{diag}\left(\lambda^{p\alpha},\ \lambda^{p\beta}\right) - J(E)\right] = 0.$$

Then

$$\lambda^{p(\alpha+\beta)} + \lambda^{p\alpha}\left(\frac{m\kappa}{1+e^{\kappa}}\right) + \lambda^{p\beta}\left(\frac{5(1+e^{\kappa}) - 4\kappa e^{\kappa}}{1+e^{\kappa}}\right) + \frac{5m\kappa}{1+e^{\kappa}} = 0,$$

With $\kappa = \Lambda/5$. Thus, we have

$$\lambda^{p(\alpha+\beta)} + \lambda^{p\alpha}\left(\frac{m\Lambda}{5\left(1+e^{\Lambda/5}\right)}\right) + \lambda^{p\beta}\left(\frac{25(1+e^{\Lambda/5}) - 4\Lambda e^{\Lambda/5}}{5\left(1+e^{\Lambda/5}\right)}\right) + \frac{m\Lambda}{1+e^{\Lambda/5}} = 0. \tag{16}$$

For a special case $\alpha = \frac{1}{p}$, $\beta = \frac{1}{p}$, the stability conditions of equilibrium point for system (11) are that the satisfy $|\arg(\lambda)| > \gamma\frac{\pi}{2}$, so (16) gives

$$\lambda^2 + \lambda\left(\frac{25(1+e^{\Lambda/5}) - 4\Lambda e^{\Lambda/5} + m\Lambda}{5\left(1+e^{\Lambda/5}\right)}\right) + \frac{m\Lambda}{1+e^{\Lambda/5}} = 0. \tag{17}$$

So $\mathrm{tr}(J) = \frac{25(1+e^{\Lambda/5}) - 4\Lambda e^{\Lambda/5} + m\Lambda}{5\left(1+e^{\Lambda/5}\right)}$, and $\det(J) = \frac{m\Lambda}{1+e^{\Lambda/5}}$.

4. Numerical Simulation

Over here we have used the Forward Euler method to see the behavior of the system (11) by varying the parameters and order of the system, so the system (11) takes the following format:

$$s^{\aleph+1} = s(0) + \frac{h^{\alpha}}{\Gamma(\alpha+1)} \sum_{\Im=0}^{\aleph} (\aleph - \Im + 1)^{\alpha}$$

$$-(\aleph - \Im)^{\alpha} \left[\Lambda - s_{\aleph+1}^{q} - 4 \frac{s_{\aleph+1}^{q} u_{\aleph+1}^{q}}{1 + \exp^{s_{\aleph+1}^{p}}} \right],$$

$$u^{\aleph+1} = u(0) + \frac{h^{\beta}}{\Gamma(\beta+1)} \sum_{\Im=0}^{\aleph} (\aleph - \Im + 1)^{\beta}$$

$$-(\aleph - \Im)^{\beta} \left[m s_{\aleph+1}^{q} \left(1 - \frac{u_{\aleph+1}^{q}}{1 + \exp^{s_{\aleph+1}^{p}}} \right) \right]$$

Case 1 Let us take $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$ $(p = 2)$, $\Lambda = 10, m = 15$ and $(s_0, u_0) = (1, 1)$. In this case we have from (16)

$$\lambda^2 + \frac{35 - 3e^2}{e^2 + 1}\lambda + \frac{150}{1 + e^2} = 0.$$

The eigenvalues from characteristic equation are:

$$\lambda_1 = -0.764\,86 - 4.\,158\,8i,\ \lambda_2 = -0.764\,86 + 4.\,158\,8i.$$

Since both real parts of the eigenvalues are negative. Also if we use the Routh–Hurwitz stability condition $(n = 2)$, it is satisfied, because $a_1 = \frac{35-3e^2}{e^2+1} > 0$ and $a_2 = \frac{150}{1+e^2} > 0$. Thus $E = \left(2, 1 + e^2\right)$ is local asymptotically stable as shown in Fig. 2.

Case 2 Let us take $\alpha = 1$, $\beta = 1$ $(p = 1)$, $\Lambda = 25, m = 1/5$ and $(s_0, u_0) = (1, 1)$. In this case we have from (16)

$$\lambda^2 - \frac{15e^5 - 6}{e^5 + 1}\lambda + \frac{5}{e^5 + 1} = 0.$$

The eigenvalues from characteristic equation are:

$\lambda_1 = 14.\,857$, $\lambda_2 = 2.\,252\,4 \times 10^{-3}$. Also if we use the Routh–Hurwitz stability condition $(n = 2)$, it is not satisfied, because $a_1 = -\frac{15e^5-6}{e^5+1} < 0$ and $a_2 = \frac{5}{e^5+1} > 0$. Since both the eigenvalues are positive.

Thus $E = \left(5, 1 + e^5\right)$ is unstable as shown in Fig. 3.

Case 3 Let us take $\alpha = 1/6$, $\beta = 1/4$ $(p = 12)$, $\Lambda = 2$, $m = 22$ and $(s_0, u_0) = (1, 1)$. In this case we have from (16)

$$\lambda^5 + (4.\,042\,1)\,\lambda^3 + (3.\,531\,5)\,\lambda^2 + 17.\,658 = 0.$$

The eigenvalues from characteristic equation are:

$\lambda_1 = -7.\,539\,1 \times 10^{-2} - 2.\,051\,6i$, $\lambda_2 = -7.\,539\,1 \times 10^{-2} + 2.\,051\,6i$, $\lambda_3 = 0.872\,10 + 1.\,367i$, $\lambda_4 = 0.872\,10 - 1.\,367i$, $\lambda_5 = -1.\,593\,4$. Then $|\arg(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_4)| > \gamma\frac{\pi}{2} = \frac{\pi}{24}$,

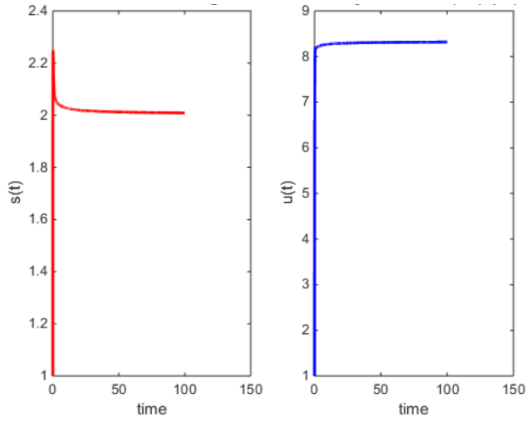

Fig. 2. Simulation of system for $\alpha = \frac{1}{2}, \beta = \frac{1}{2}, \Lambda = 10, m = 15$

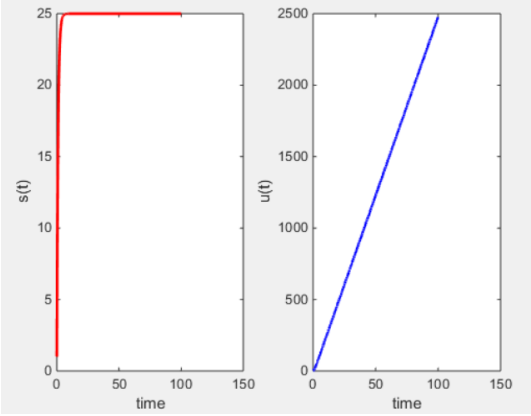

Fig. 3. Simulation of system for $\alpha = 1, \beta = 1, \Lambda = 25, m = 1/5$

When

$$
\begin{aligned}
|\arg(\lambda_1)| &= \left|\tan^{-1}\left(\frac{-2.0516}{-7.5391\times 10^{-2}}\right)\right| = 1.534 > \frac{\pi}{24},\\
|\arg(\lambda_2)| &= \left|\tan^{-1}\left(\frac{2.0516}{-7.5391\times 10^{-2}}\right)\right| = 1.534 > \frac{\pi}{24},\\
|\arg(\lambda_3)| &= \left|\tan^{-1}\left(\frac{1.367}{0.8721}\right)\right| = 1.0029 > \frac{\pi}{24},\\
|\arg(\lambda_4)| &= \left|\tan^{-1}\left(\frac{-1.367}{0.8721}\right)\right| = 1.0029 > \frac{\pi}{24},\\
|\arg(\lambda_5)| &= \left|\tan^{-1}\left(\frac{0}{-1.5934}\right)\right| = \pi > \frac{\pi}{24}.
\end{aligned}
$$

Thus, $E = (0.4, 1 + e^{0.4})$ is asymptotically stable as shown in Fig. 4.

5. Conclusion

In this work, we studied a nonlinear mathematical model for the dynamics of the fractional-order Lengyel-Epstein chemical reaction, a well-known example of oscillatory chemical behavior. By incorporating fractional derivatives, Through numerical simulations using Forward Euler method, we observed that the solutions consistently converge to the system's equilibrium point.

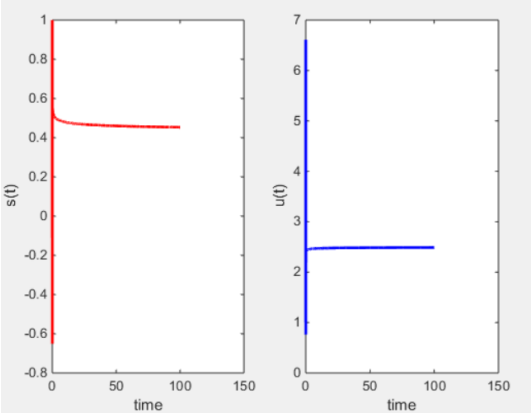

Fig. 4. Simulation of system for $\alpha = 1/6$, $\beta = 1/4, \Lambda = 2, m = 22$

However, the trajectories vary markedly with different values of the fractional order α and β, highlighting the significant influence of fractional dynamics on the transient behavior of the system. These results underscore the value of fractional order modeling in capturing the fine dynamics of oscillating chemical reactions and pave the way for further exploration into the stability and control of these systems.

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