IJRCS
International Journal of Robotics and Control Systems
Vol. 5, No. 2, 2025, pp. 1596-1607
ISSN 2775-2658
http://pubs2.ascee.org/index.php/ijrcs
ASCEE

Analytical Investigation of the Existence and Ulam Stability of Integro-Differential Equations with Conformable Derivatives Under Non-Local Conditions

Seddiki Fakhreddine [a,1], Ayman A Hazaymeh [b,2], Mazin Aljazzazi [c,3], Reham Qaralleh [d,4], Anwar Bataihah [b,5], Iqbal M. Batiha [e,f,6,*], Rasha Ibrahim Hajaj [c,7]

[a] Department of Mathematics, University of Ziane Achour, Djelfa 17000, Algeria
[b] Department of Mathematics, Faculty of Science, Jadara University, Irbid 21110, Jordan
[c] Department of Mathematics, University of Jordan, Amman, Jordan
[d] Department of Scientific Basic Sciences, Middle East University, Amman, Jordan
[e] Department of Mathematics, Al Zaytoonah University of Jordan, Amman 11733, Jordan
[f] Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman 346, United Arab Emirates
[1] fakhreddine.seddiki@univ-djelfa.dz; [2] aymanha@jadara.edu.jo; [3] m.jazazi@yahoo.com;
[4] r.qaralleh@meu.edu.jo; [5] anwerbataihah@gmail.com; [6] i.batiha@zuj.edu.jo; [7] rasha_hejaj@yahoo.com
*Corresponding Author

ARTICLE INFO

Article History
Received March 12, 2025
Revised June 24, 2025
Accepted July 07, 2025

Keywords
Conformable Fractional Derivative;
Integro-Differential Equation;
Banach Fixed-Point Theorem;
Krasnoselskii Fixed-Point Theorem;
Ulam Stability

ABSTRACT

This study examines an integro-differential equation involving fractional conformable derivatives and non-local conditions. It proves the existence and uniqueness of mild solutions by applying the Banach fixed-point theorem. Furthermore, it demonstrates a notable result about the existence of at least one solution, backed by conditions based on the Krasnoselskii fixed-point theorem. The investigation also explores the Ulam stability of integro-differential equations. To highlight the practical relevance and robustness of the findings, an illustrative example is provided.

This is an open access article under the CC-BY-SA license.

1. Introduction

The fractional derivative extends the concept of the classical derivative to an arbitrary order and has been a part of calculus for centuries. Over the past thirty years, it has found applications across a wide range of fields, including science, engineering, and mathematics [1]–[7]. There are various definitions of fractional derivatives in the literature today, each relying on different assumptions. Among the most widely recognized are the Riemann–Liouville and Caputo fractional derivatives, see [8]–[13]. However, it's important to highlight that these two forms of derivatives do not adhere to the classical chain rule.

In [14], Khalil proposed an operator derived from a local limit, which serves as a natural extension of the traditional derivative while maintaining the core properties of the classical derivative. This operator is known as the conformable fractional derivative [15], [16]. In [17], Abdeljawad further examined this derivative, addressing topics such as the chain rule, mean value theorem, Grönwall inequality, exponential function, and Laplace transform. Its various applications and extensions were also explored [18]–[26].

The existence of solutions to initial value problems for certain conformable fractional differential equations has been extensively explored in the literature, particularly in studies [27]–[33]. These works aim to identify the conditions that guarantee the existence of solutions, examine the properties of these solutions, and provide the theoretical framework that underpins their existence. Additionally, several research papers have been dedicated to studying the Hyers–Ulam stability of conformable differential equations [34]–[37]. This concept, which deals with the stability of solutions to functional equations under small perturbations, has been applied to a wide range of conformable fractional differential equations.

In this paper, we investigate the existence and stability of the conformable fractional integro-differential equations with non-local conditions in Banach spaces $(U, \|\cdot\|_U)$. A conformable integro-differential equation with non-local conditions can be written as:

$$\psi^{(\alpha)}(t) = \phi(t, \psi(t), N\psi(t)), \quad t \in [0, T], \quad 0 < \alpha \leq 1,$$

With non-local conditions of the form:

$$\psi(0) = \psi_0 + \varphi(\psi),$$

Where:
- $\psi^{(\alpha)}(t)$ denotes the conformable fractional derivative of order α.
- $N\psi(t)$ is an integral operator given by:

$$N\psi(t) = \int_0^t f(s, \psi(s))\, ds,$$

Where $f \in C(I \times U, U)$ is a given function.
- $\phi \in C(I \times U \times U, U)$ is a nonlinear function.
- $\varphi(\psi)$ represents a non-local condition, which means the solution at a point depends on its values over an entire interval rather than just an initial condition.

The remainder of the paper is structured as follows. Section 2 presents key definitions and theorems that are essential for the subsequent analysis. In Section 3, we first establish criteria for the existence of solutions to the non-local problem using fixed point theorems and the conformable fractional calculus. We also further explore the Ulam stability of these solutions.

2. Preliminaries

Khalil *et al.* proposed a completely innovative definition of fractional calculus, which is formulated as follows:

Definition 2.1 [14]Let $\psi : [0, \infty) \to \mathbb{R}$ be a real valued function. Then the conformable fractional derivative of ψ of order $\alpha \in (0, 1]$, at $t > 0$ is defined by

$$\psi^{(\alpha)}(t) = \lim_{\epsilon \to 0} \frac{\psi(t + \epsilon t^{1-\alpha}) - \psi(t)}{\epsilon},$$

and

$$\psi^{(\alpha)}(0) = \lim_{t \to 0^+} \psi^{(\alpha)}(t).$$

When the limit exists, we say that ψ is α-differentiable at t.
The integral operator corresponding to the derivative operator is given by the following definition.

$$I_\alpha^a(\psi)(t) = \int_a^t \frac{\psi(s)}{s^{1-\alpha}}\, ds.$$

Theorem 2.2 [14] Let $\alpha \in (0, 1]$ and ψ, ω be α-differentiable at a point $t > 0$ then

1. $(\psi + \omega)^{(\alpha)}(t) = a\psi^{(\alpha)}(t) + b\omega^{(\alpha)}(t)$ for all $a, b \in\mathbb{R}$.
2. $(t^p)^{(\alpha)}(t) = pt^{t^{p-\alpha}} for\ all\ p \in\mathbb{R}$.
3. $\psi^{(\alpha)}(t) = 0$, for all constant function $\psi(t) = \lambda$.
4. $(\psi\omega)^{(\alpha)} = \psi\omega^{(\alpha)} + \omega\psi^{(\alpha)}$.
5. In addition, ψ is differentiable then $\psi^{(\alpha)}(t) = t^{1-\alpha}\frac{d\psi}{dt}(t)$.

Theorem 2.3 [17] Let $0 < \alpha \leq 1$, and $\psi : [a, +\infty) \to \mathbb{R}$ be a function, where $a \geq 0$. For $t \geq a$, we have if ψ is a differentiable function, then $I^a_\alpha\psi^{(\alpha)}(t) = \psi(t) - \psi(a)$.

Theorem 2.4 [38]**(Krasnoselskii Fixed-Point Theorem)**. Let M be a closed, convex, and non-empty subset of a Banach space U, and let $P : M \to M$ be a mapping of the form:

$$Pz = Bz + Az,$$

Where:

- A is a continuous and compact operator,
- B is a contraction, i.e., there exists a constant $k \in [0, 1)$ such that

$$\|B(x) - B(y)\| \leq k\|x - y\|$$

for all $x, y \in M$
Then, P has at least one fixed point in M.

Theorem 2.5 **(Banach Fixed Point Theorem)**. Let $(U, \|\cdot\|)$ be a Banach space, and let $P : U \to U$ be a contraction mapping. Then, P has a unique fixed point.

3. Main Results

Let $C(I, U)$ be the space of functions $\psi(\cdot)$ defined on $I = [0, T]$ with values in U, such that $\psi(\cdot)$ is continuous on the interval I. It is evident that $C(I, U)$ forms a Banach space with the norm

$$\|\psi\| = \sup\{\|\psi(t)\|_U : t \in I\}.$$

Additionally, the space $C^\alpha(I, U)$ consists of functions $\psi : I \to U$ that are α-continuously differentiable, which generalizes the notion of differentiability to fractional orders. We present the expression for the mild solution of the problem

$$\begin{cases} \psi^{(\alpha)}(t) = \phi(t, \psi(t), N\psi(t)), \quad t \in [0, T], \quad 0 < \alpha \leq 1, \\ \psi(0) = \psi_0 + \varphi(\psi). \end{cases} \tag{1}$$

Lemma 3.1 A solution $\psi \in C(I, U)$ of problem (1) has the following form

$$\psi(t) = \psi_0 + \varphi(\psi) + \int_0^t s^{\alpha-1}\phi(s, \psi(s), N\psi(s))\, ds, \quad t \in I.$$

Proof 1 Applying the conformable fractional integral of order α to both sides of the differential equation, we use the fundamental property of conformable fractional derivatives:

$$I^{\alpha}\psi^{(\alpha)}(t) = \psi(t) - \psi(0).$$

Thus, integrating both sides from 0 to t, we obtain:

$$\psi(t) = \psi(0) + \int_0^t s^{\alpha-1}\phi(s, \psi(s), N\psi(s))\,ds.$$

Substituting the initial condition $\psi(0) = \psi_0 + \varphi(\psi)$, we get:

$$\psi(t) = \psi_0 + \varphi(\psi) + \int_0^t s^{\alpha-1}\phi(s, \psi(s), N\psi(s))\,ds.$$

Definition 3.2 A function $\psi \in C(I, U)$ is called a mild solution of problem (1) if it satisfies the integral equation

$$\psi(t) = \psi_0 + \varphi(\psi) + \int_0^t s^{\alpha-1}\phi(s, \psi(s), N\psi(s))\,ds, \quad t \in I. \tag{2}$$

3.1. Existence of Mild Solutions

Our objective is to explore the existence theory for the conformable integro-differential equation with non-local conditions, making use of fundamental concepts. Our study is based on the functional analysis theorems of Banach and Krasnoselskii. For any $\psi \in C(I, U)$, let

$$\digamma\psi(t) = \psi_0 + \varphi(\psi) + \int_0^t s^{\alpha-1}\phi(s, \psi(s), N\psi(s))\,ds. \tag{3}$$

It is easy to show that $\digamma\psi : C(I, U) \to C(I, U)$, and the problem (1) is equivalent to the integral equation (2), that is, $\psi = \psi(t)$ is a solution of (1) if and only if $\psi \in C(I, U)$ is a fixed point of the operator $\digamma$.

Let us present the assumptions required to ensure the uniqueness of the solution for our results in this section.

(A1) There exist constants $\delta_1, \delta_2 > 0$ such that

$$\|\phi(t, \psi, \omega) - \phi(t, \overline{\psi}, \overline{\omega})\| \le \delta_1\|\psi - \overline{\psi}\| + \delta_2\|\omega - \overline{\omega}\|, \quad \forall \psi, \omega, \overline{\psi}, \overline{\omega} \in C(I, U).$$

(A2) There exists constant $\gamma > 0$ such that

$$\|f(t, \psi) - f(t, \overline{\psi})\| \le \gamma\|\psi - \overline{\psi}\|, \quad \forall \psi, \overline{\psi} \in C(I, U).$$

(A3) There exists constant $\eta > 0$ such that

$$\|\varphi(\psi) - \varphi(\overline{\psi})\| \le \eta\|\psi - \overline{\psi}\|, \quad \forall \psi, \overline{\psi} \in C(I, U).$$

Theorem 3.3 Suppose (A1)-(A3) hold. If

$$\eta + \delta_1\frac{T^{\alpha}}{\alpha} + \gamma\delta_2\frac{T^{\alpha+1}}{\alpha} < 1 \tag{4}$$

holds, then the conformable integro-differential equation (1) has a unique solution.

Proof 2 For any $\psi, \overline{\psi} \in C(I,U)$, and any $t \in I$, from assumptions (A1) and (A2), we get

$$\begin{aligned}\|\phi(t,\psi,N\psi)-\phi(t,\overline{\psi},N\overline{\psi})\| &\leq \delta_1\|\psi-\overline{\psi}\|+\delta_2\|N\psi-N\overline{\psi}\| \\ &\leq \delta_1\|\psi-\overline{\psi}\|+\delta_2\int_0^t\|f(s,\psi(s))-f(s,\overline{\psi}(s))\|ds \\ &\leq \delta_1\|\psi-\overline{\psi}\|+\delta_2\int_0^t\gamma\|\psi-\overline{\psi}\|ds \\ &\leq \delta_1\|\psi-\overline{\psi}\|+\delta_2\gamma T\|\psi-\overline{\psi}\|.\end{aligned}$$

Thus, we have

$$\begin{aligned}\|F(\psi)-F(\overline{\psi})\| &\leq \|\varphi(\psi)-\varphi(\overline{\psi})\|+\int_0^t s^{\alpha-1}\left(\delta_1\|\psi-\overline{\psi}\|+\delta_2\gamma T\|\psi-\overline{\psi}\|\right)ds \\ &\leq \eta\|\psi-\overline{\psi}\|+\left(\delta_1\frac{T^\alpha}{\alpha}+\gamma\delta_2\frac{T^{\alpha+1}}{\alpha}\right)\|\psi-\overline{\psi}\| \\ &\leq \left(\eta+\delta_1\frac{T^\alpha}{\alpha}+\gamma\delta_2\frac{T^{\alpha+1}}{\alpha}\right)\|\psi-\overline{\psi}\|.\end{aligned}$$

By inequality (4), it follows that the operator F is contractive. Hence, by the Banach fixed-point theorem, F admits a unique fixed point in $C(I,U)$. Consequently, the conformable fractional integro-differential equations with non-local conditions (1) has a unique solution.

We will now apply Krasnoselskii's fixed point theorem to prove an existence result. To proceed, we introduce the following hypotheses:
(H1) There exist continuous functions $\rho, \mu, \tau, \sigma, \theta$ mapping I into $[0,+\infty)$ such that, for all $\psi, \overline{\psi} \in C(I,U)$ and $t \in I$, the following inequalities hold:

$$\|\phi(t,\psi,\overline{\psi})\| \leq \rho(t)+\mu(t)\|\psi\|+\tau(t)\|\overline{\psi}\|, \quad \|f(t,\psi)\| \leq \sigma(t)+\theta(t)\|\psi\|.$$

Furthermore, we define the supremum values:

$$\rho^*=\sup_{t\in I}\rho(t), \quad \mu^*=\sup_{t\in I}\mu(t), \quad \tau^*=\sup_{t\in I}\tau(t), \quad \sigma^*=\sup_{t\in I}\sigma(t), \quad \theta^*=\sup_{t\in I}\theta(t).$$

(H2) There exists a positive constant ℓ^* in $(0,1)$ and a nonnegative, nondecreasing function ν in $C([0,\infty))$ such that

$$\nu(z) < \ell^* z, \quad \text{for } z>0,$$

and

$$\|\varphi(\psi)-\varphi(\overline{\psi})\| \leq \nu(\|\psi-\overline{\psi}\|),$$

for any $\psi, \overline{\psi} \in C(I,U)$ and $\varphi(0)=0$.
(H3) Suppose that

$$\beta=\ell^*+\frac{\mu^*T^\alpha}{\alpha}+\frac{\tau^*\theta^*T^{\alpha+1}}{\alpha}<1.$$

Let $B_\xi=\{\psi \in C(I,U) : \|\psi\| \leq \xi \quad \text{and} \quad \xi \geq \frac{\beta^*}{1-\beta}\}$ and $\beta^*=\|\psi_0\|+\frac{\rho^*T^\alpha}{\alpha}+\frac{\tau^*\sigma^*T^{\alpha+1}}{\alpha}$. Then B_ξ is clearly a bounded closed and convex subset in $C(I,U)$. Consider the operator F defined in (3) on B_ξ , and divide it into two operators Λ and Υ on B_ξ as follows:

$$\begin{aligned}\Lambda(\psi(t)) &= \psi_0+\varphi(\psi), \\ \Upsilon(\psi(t)) &= \int_0^t s^{\alpha-1}\phi(s,\psi(s),N\psi(s))\,ds.\end{aligned}$$

Theorem 3.4 Given that the hypotheses (H1), (H2), and (H3) are satisfied, problem (1) admits at least one solution $\psi \in B_\xi$.

Proof 3 The proof consists of four steps:
Step 1: We demonstrate that $(\Lambda + \Upsilon)(B_\xi) \subseteq B_\xi$.
For any $t \in I$ and $\psi \in B_\xi$, we have

$$\begin{aligned}\|\Lambda(\psi) + \Upsilon(\psi)\| &\leq \|\psi_0\| + \frac{\rho^* T^\alpha}{\alpha} + \frac{\tau^*\sigma^* T^{\alpha+1}}{\alpha} + \left(\ell^* + \frac{\mu^* T^\alpha}{\alpha} + \frac{\tau^*\theta^* T^{\alpha+1}}{\alpha}\right)\|\psi\| \\ &\leq \beta^* + \beta\xi \\ &\leq \xi.\end{aligned}$$

Thus, $(A + P)(B_\xi) \subseteq B_\xi$.
Step 2: We show that Λ is a contraction mapping.
For any $\psi, \overline{\psi} \in B_\xi$, we get

$$\begin{aligned}\|\Lambda(\psi) - \Lambda(\overline{\psi})\| = \|\varphi(\psi) - \varphi(\overline{\psi})\| &\leq \nu(\|\psi - \overline{\psi}\|) \\ &\leq \ell^*\|\psi - \overline{\psi}\|.\end{aligned}$$

Since $\ell^* < 1$, then Λ is a contraction mapping.
Step 3: We will prove that the operator Υ is continuous.
Let $\{\psi_n\}$ be a sequence such that $\psi_n \to \psi$ in B_ξ. Define $\phi_n(\cdot) = \phi(\cdot, \psi_n, N\psi_n)$ and $\phi(\cdot) = \phi(\cdot, \psi, N\psi)$. We note that

$$s^{\alpha-1}\phi_n(s) \to s^{\alpha-1}\phi(s) \quad \text{as } n \to \infty,$$

and the bound

$$s^{\alpha-1}\|\phi_n - \phi\| \leq 2s^{\alpha-1}\left(\rho^* + \tau^*\sigma^* T + \xi(\mu^* + \tau^*\theta^* T)\right)$$

holds. Applying the Lebesgue Dominated Convergence Theorem, we obtain the inequality

$$\|\Upsilon\psi_n(t) - \Upsilon\psi(t)\| \leq \int_0^t s^{\alpha-1}\|\phi_n - \phi\| ds, \quad t \in I,$$

which tends to zero as $n \to \infty$. Therefore, the operator Υ is continuous.
Step 4. We show $\Upsilon(B_\xi)$ is equicontinuous.
For $\psi \in B_\xi$ and $t_1, t_2 \in I$ such that $t_1 < t_2$, we have

$$\begin{aligned}\|\Upsilon\psi(t_2) - \Upsilon\psi(t_1)\| &\leq (\rho^* + \tau^*\sigma^* T + \xi(\mu^* + \tau^*\theta^* T)) \int_{t_2}^{t_1} s^{\alpha-1} ds \\ &= (\rho^* + \tau^*\sigma^* T + \xi(\mu^* + \tau^*\theta^* T)) \left(\frac{t_2^\alpha - t_1^\alpha}{\alpha}\right).\end{aligned}$$

Thus, $\|\Upsilon\psi(t_2) - \Upsilon\psi(t_1)\| \to 0$ as $t_1 \to t_2$. Therefore, $\Upsilon(B_\xi)$ is equicontinuous.
From the Arzela–Ascoli theorem, $\Upsilon(B_\xi)$ is relatively compact. Finally, the Krasnoselskii fixed point theorem completes the proof.

3.2. Ulam Stability Analysis

Ulam stability analysis examines whether small perturbations in functional equations allow approximate solutions to be adjusted into exact ones, ensuring stability and resilience in mathematical structures. This section explores the concepts of Ulam–Hyers stability and Ulam–Hyers–Rassias stability.

Ulam–Hyers stability

Definition 3.5 The equation (1) is said to be Ulam–Hyers stable if, for every approximate solution $\tilde{\psi}(t)$ satisfying

$$\left\| \tilde{\psi}^{(\alpha)}(t) - \phi(t, \tilde{\psi}(t), N\tilde{\psi}(t)) \right\| \leq \epsilon, \quad \forall t \in I, \tag{5}$$

there exists an exact solution $\psi(t)$ of (1) such that

$$\|\tilde{\psi} - \psi\| \leq C\epsilon, \quad \forall t \in I, \tag{6}$$

Where C is a constant independent of ϵ.

Remark 3.6 A function $\tilde{\psi} \in C(I, U)$ is a solution of (5) if and only if there exists a function $g \in C(I, U)$ such that:

(i) $\|g(t)\| \leq \epsilon$, for all $t \in I$,

(ii) $\tilde{\psi}^{(\alpha)}(t) = \phi(t, \tilde{\psi}(t), N\tilde{\psi}(t)) + g(t)$, for all $t \in I$.

Lemma 3.7 Let $\tilde{\psi} \in C(I, U)$ be a solution of (5). Then $\tilde{\psi}$ satisfies the inequality:

$$\left\| \tilde{\psi}(t) - \psi_0 - \varphi(\tilde{\psi}) - \int_0^t s^{\alpha-1} \phi(s, \tilde{\psi}(s), N\tilde{\psi}(s))\, ds \right\| \leq \epsilon \frac{T^\alpha}{\alpha}, \quad t \in I. \tag{7}$$

Proof 4 From Remark 3.6, the solution of the equation

$$\tilde{\psi}^{(\alpha)}(t) = \phi(t, \tilde{\psi}(t), N\tilde{\psi}(t)) + g(t)$$

can be expressed as

$$\tilde{\psi}(t) = \psi_0 + \varphi(\tilde{\psi}) + \int_0^t s^{\alpha-1} \phi(s, \tilde{\psi}(s), N\tilde{\psi}(s))\, ds + \int_0^t s^{\alpha-1} g(s) ds.$$

For all $t \in I$, we obtain

$$\left\| \tilde{\psi}(t) - \psi_0 - \varphi(\tilde{\psi}) - \int_0^t s^{\alpha-1} \phi(s, \tilde{\psi}(s), N\tilde{\psi}(s))\, ds \right\| \leq \int_0^t s^{\alpha-1} \|g(s)\| ds \leq \epsilon \frac{T^\alpha}{\alpha}.$$

Theorem 3.8 If all the conditions of Theorem 3.3 hold, then the conformable fractional integro-differential equation with non-local conditions (1) is U-H stable.

Proof 5 As all the conditions of Theorem 3.3 are fulfilled, the conformable fractional integro-differential equation with non-local conditions (1) has a unique solution $\psi \in C(I, U)$.

Let $\tilde{\psi} \in C(I, U)$ be a solution of the inequality system (5). According to Lemma 3.7 and Remark 3.6, for any $t \in I$, we have:

$$\begin{aligned} \|\tilde{\psi}(t) - \psi(t)\| &= \|\tilde{\psi}(t) - \psi_0 - \varphi(\psi) - \int_0^t s^{\alpha-1} \phi(s, \psi(s), N\psi(s))\, ds\| \\ &\leq \left\| \tilde{\psi}(t) - \psi_0 - \varphi(\tilde{\psi}) - \int_0^t s^{\alpha-1} \phi(s, \tilde{\psi}(s), N\tilde{\psi}(s))\, ds \right\| + \|\varphi(\tilde{\psi}) - \varphi(\psi)\| \\ &+ \left\| \int_0^t s^{\alpha-1} (\phi(s, \tilde{\psi}(s), N\tilde{\psi}(s)) - \phi(s, \psi(s), N\psi(s))\, ds \right\| \\ &\leq \epsilon \frac{T^\alpha}{\alpha} + \left(\eta + \delta_1 \frac{T^\alpha}{\alpha} + \gamma \delta_2 \frac{T^{\alpha+1}}{\alpha} \right) \|\tilde{\psi} - \psi\|. \end{aligned}$$

Hence, we obtain

$$\|\tilde{\psi} - \psi\| \leq \frac{T^{\alpha}}{\alpha(1-\varpi)}\epsilon,$$

Where $\varpi = \left(\eta + \delta_1 \frac{T^{\alpha}}{\alpha} + \gamma\delta_2 \frac{T^{\alpha+1}}{\alpha}\right)$. By Definition 3.5, the conformable fractional integro-differential equation with non-local conditions (1) is U-H stable.

Ulam–Hyers–Rassias Stability

Definition 3.9 The equation (1) is said to be Ulam–Hyers–Rassias stable if, for every approximate solution $\tilde{\psi}(t)$ satisfying

$$\left\|\tilde{\psi}^{(\alpha)}(t) - \phi(t, \tilde{\psi}(t), N\tilde{\psi}(t))\right\| \leq \epsilon h(t), \quad \forall t \in I, \tag{8}$$

Where $h : I \to \mathbb{R}^+$ is a given function, there exists an exact solution $\psi(t)$ of (1) such that

$$\|\tilde{\psi} - \psi\| \leq C\epsilon h(t), \quad \forall t \in I, \tag{9}$$

Where C is a constant independent of ϵ.

Remark 3.10 A function $\tilde{\psi} \in C(I, U)$ is a solution of (8) if and only if there exists a function $\widetilde{g} \in C(I, U)$ such that:

(i) $\|\widetilde{g}(t)\| \leq \epsilon h(t)$, for all $t \in I$,

(ii) $\tilde{\psi}^{(\alpha)}(t) = \phi(t, \tilde{\psi}(t), N\tilde{\psi}(t)) + \widetilde{g}(t)$, for all $t \in I$.

Lemma 3.11 Let $\tilde{\psi} \in C(I, U)$ be a solution of (8). Then $\tilde{\psi}$ satisfies the inequality for all $t \in I$:

$$\left\|\tilde{\psi}(t) - \psi_0 - \varphi(\tilde{\psi}) - \int_0^t s^{\alpha-1}\phi(s, \tilde{\psi}(s), N\tilde{\psi}(s))\, ds\right\| \leq \epsilon \int_0^t s^{\alpha-1}h(s)ds. \tag{10}$$

Proof 6 From Remark 3.10, the solution of the equation

$$\tilde{\psi}^{(\alpha)}(t) = \phi(t, \tilde{\psi}(t), N\tilde{\psi}(t)) + \widetilde{g}(t)$$

can be expressed as

$$\tilde{\psi}(t) = \psi_0 + \varphi(\tilde{\psi}) + \int_0^t s^{\alpha-1}\phi(s, \tilde{\psi}(s), N\tilde{\psi}(s))\, ds + \int_0^t s^{\alpha-1}\widetilde{g}(s)ds.$$

For all $t \in I$, we get

$$\left\|\tilde{\psi}(t) - \psi_0 - \varphi(\tilde{\psi}) - \int_0^t s^{\alpha-1}\phi(s, \tilde{\psi}(s), N\tilde{\psi}(s))\, ds\right\| \leq \int_0^t s^{\alpha-1}\|\widetilde{g}(s)\|ds \leq \epsilon \int_0^t s^{\alpha-1}h(s)ds.$$

Theorem 3.12 Suppose that all the assumptions of Theorem 3.3 are satisfied. Additionally, assume that there exists a continuous function $h(t) > 0$ for $t \in I$ and a constant $c_h > 0$ such that the following inequality holds:

$$\int_0^t s^{\alpha-1}h(s)\, ds \leq c_h h(t), \quad \forall t \in I.$$

Then, the conformable fractional integro-differential equation with non-local conditions (1) is U-H-R stable.

Proof 7 Since all the conditions of Theorem 3.3 are satisfied, the conformable fractional integro-differential equation with non-local conditions (1) admits a unique solution $\psi \in C(I, U)$.

Let $\tilde{\psi} \in C(I, U)$ be a solution of the inequality system (8). Based on Lemma 3.11 and Remark 3.10, it follows that for any $t \in I$, we have:

$$\begin{aligned}
\|\tilde{\psi}(t) - \psi(t)\| &= \|\tilde{\psi}(t) - \psi_0 - \varphi(\psi) - \int_0^t s^{\alpha-1}\phi(s, \psi(s), N\psi(s))\, ds\| \\
&\leq \left\|\tilde{\psi}(t) - \psi_0 - \varphi(\tilde{\psi}) - \int_0^t s^{\alpha-1}\phi(s, \tilde{\psi}(s), N\tilde{\psi}(s))\, ds\right\| + \|\varphi(\tilde{\psi}) - \varphi(\psi)\| \\
&+ \left\|\int_0^t s^{\alpha-1}(\phi(s, \tilde{\psi}(s), N\tilde{\psi}(s)) - \phi(s, \psi(s), N\psi(s))\, ds\right\| \\
&\leq \epsilon \int_0^t s^{\alpha-1}h(s)ds + \left(\eta + \delta_1\frac{T^\alpha}{\alpha} + \gamma\delta_2\frac{T^{\alpha+1}}{\alpha}\right)\|\tilde{\psi} - \psi\| \\
&\leq \epsilon c_h h(t) + \left(\eta + \delta_1\frac{T^\alpha}{\alpha} + \gamma\delta_2\frac{T^{\alpha+1}}{\alpha}\right)\|\tilde{\psi} - \psi\|.
\end{aligned}$$

Thus, we get

$$\|\tilde{\psi} - \psi\| \leq \frac{c_h}{1 - \varpi}\epsilon h(t),$$

Where $\varpi = \left(\eta + \delta_1\frac{T^\alpha}{\alpha} + \gamma\delta_2\frac{T^{\alpha+1}}{\alpha}\right)$. By Definition 3.9, the conformable fractional integro-differential equation with non-local conditions (1) is U-H-R stable.

Example 3.13 We examine the following conformable integro-differential equation subject to non-local conditions in $\mathbb{R}$:

$$\begin{cases}\psi^{(\frac{1}{2})}(t) = \frac{1}{4}\left(t + \frac{\psi(t)}{t+1}\right) + \int_0^t \frac{\cos(s)}{4}\psi(s)ds, \quad t \in [0, 1], \\ \psi(0) = \psi_0 + \int_0^1 \frac{1}{3}\psi(s)ds.\end{cases} \tag{11}$$

Corresponding to equation (11), where

$$\alpha = \frac{1}{2}, \quad \varphi(\psi) = \int_0^1 \frac{1}{3}\psi(s)ds,$$

and

$$\phi(t, \psi(t), N\psi(t)) = \frac{1}{4}\left(t + \frac{\psi(t)}{t+1}\right) + \int_0^t \frac{\cos(s)}{4}\psi(s)ds, \quad f(t, \psi(t)) = \frac{\cos(t)}{4}\psi(t).$$

For any $\psi, \omega, \overline{\psi}, \overline{\omega} \in C(I, \mathbb{R})$, and any $t \in [0, 1]$, we have

$$\|f(t, \psi) - f(t, \overline{\psi})\| \leq \frac{1}{4}\|\psi - \overline{\psi}\|,$$

$$\|\phi(t, \psi, \omega) - \phi(t, \overline{\psi}, \overline{\omega})\| \leq \frac{1}{4}\|\psi - \overline{\psi}\| + \frac{1}{4}\|\omega - \overline{\omega}\|,$$

and

$$\|\varphi(\psi) - \varphi(\overline{\psi})\| \leq \frac{1}{3}\|\psi - \overline{\psi}\|.$$

It is clear that $\eta + \delta_1\frac{T^\alpha}{\alpha} + \gamma\delta_2\frac{T^{\alpha+1}}{\alpha} = \frac{23}{24} < 1$. According to Theorem 3.3, the conformable fractional integro-differential equation with non-local conditions (11) possesses a unique solution.

Moreover, Theorem 3.8 ensures that conformable fractional integro-differential equation (11) is U-H stable.
Let $h(t) = \sqrt{t} + 1, t \in [0, 1]$, we have

$$\int_0^t s^{-\frac{1}{2}} (\sqrt{s} + 1)\, ds = t + 2\sqrt{t} \le 3 \le 3(\sqrt{t} + 1) = 3h(t),$$

where $c_h = 3$. All the conditions of Theorem 3.12 are satisfied, then the conformable fractional integro-differential equation with non-local conditions (11) is U-H-R stable.

4. Conclusion

This paper investigates a class of conformable fractional integro-differential equations with non-local conditions. After addressing the existence and uniqueness of solutions, we concentrate on the Ulam-Hyers (U-H) stability and Ulam-Hyers-Rassias (U-H-R) stability of the solutions to problem (1). Finally, an example is presented to illustrate the practicality and effectiveness of our results. In our future work, we aim to extend the current research by investigating more general forms of conformable fractional integro-differential equations.

Author Contribution: All authors contributed equally to the main contributor to this paper. All authors read and approved the final paper.

Funding: This research received no external funding.

Acknowledgment: The authors sincerely thank the reviewers for their insightful comments and valuable suggestions, which have significantly improved the quality of this manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

References

[1] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, 2000, https://books.google.co.id/books?id=MIXVCgAAQBAJ&hl=id&source=gbs_navlinks_s.

[2] G. Barbero *et al.*, "A Brief Review of Fractional Calculus as a Tool for Applications in Physics: Adsorption Phenomena and Electrical Impedance in Complex Fluids," *Fractal and Fractional*, vol. 8, no. 7, p. 369, 2024, https://doi.org/10.3390/fractalfract8070369.

[3] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, 2006, https://books.google.co.id/books?id=uxANOU0H8IUC&hl=id&source=gbs_navlinks_s.

[4] E. C. de Oliveira and J. A. Tenreiro Machado, "A review of definitions for fractional derivatives and integrals," *Mathematical Problems in Engineering*, vol. 2014, pp. 1–6, 2014, http://dx.doi.org/10.1155/2014/238459.

[5] G. Farraj, B. Maayah, R. Khalil, and W. Beghami, "An algorithm for solving fractional differential equations using conformable optimized decomposition method," *International Journal of Advances in Soft Computing and Its Applications*, vol. 15, no. 1, 2023, https://doi.org/10.15849/IJASCA.230320.13.

[6] M. Berir, "Analysis of the effect of white noise on the Halvorsen system of variable-order fractional derivatives using a novel numerical method," *International Journal of Advances in Soft Computing and its Applications*, vol. 16, no. 3, pp. 294–306, 2024, https://doi.org/10.15849/IJASCA.241130.16.

[7] N. R. Anakira *et al.*, "An algorithm for solving linear and non-linear Volterra Integro-differential equations," *International Journal of Advances in Soft Computing & Its Applications*, vol. 15, no. 3, pp. 77–83, 2023, https://doi.org/10.15849/IJASCA.231130.05.

[8] I. M. Batiha, H. F. Abdalsmad, I. H. Jebril, H. O. Al-Khawaldeh, W. A. A. AlKasasbeh, and S. Momani, "Trapezoidal scheme for the numerical solution of fractional initial value problems," *International Journal of Robotics and Control Systems*, vol. 5, no. 2, pp. 1238–1253, 2025, http://dx.doi.org/10.31763/ijrcs.v5i2.1795.

[9] N. Djeddi, I. Batiha, N. Harrouche, M. Al-Smadi, and S. Momani, "Advanced solutions for nonlinear fractional equations: a Laplace–Caputo–RKDM approach," *Gulf Journal of Mathematics*, vol. 19, no. 2, pp. 93–110, 2025, https://doi.org/10.56947/gjom.v19i2.2575.

[10] A. Boudjedour, I. Batiha, S. Boucetta, M. Dalah, K. Zennir, and A. Ouannas, "A finite difference method on uniform meshes for solving the time-space fractional advection–diffusion equation," *Gulf Journal of Mathematics*, vol. 19, no. 1, pp. 156–168, 2025, https://doi.org/10.56947/gjom.v19i1.2524.

[11] S. Alshorm and I. M. Batiha, "Stochastic population growth model using three-point fractional formula," in *Mathematical Analysis and Numerical Methods*, pp. 457–465, 2024, https://doi.org/10.1007/978-981-97-4876-1_31.

[12] M. Shqair *et al*., "Numerical solutions for fractional multi-group neutron diffusion system of equations," *International Journal of Neutrosophic Science*, vol. 24, no. 4, pp. 8–38, 2024, https://doi.org/10.54216/IJNS.240401.

[13] I. Batiha, A. Abdelnebi, M. Shqair, I. H. Jebril, S. Alkhazaleh, and S. Momani, "Computational approaches to two-energy group neutron diffusion in cylindrical reactors," *Journal of Robotics and Control (JRC)*, vol. 5, no. 6, pp. 1980–1989, 2024, https://doi.org/0.18196/jrc.v5i6.23392.

[14] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, "A new definition of fractional derivative," *Journal of Computational and Applied Mathematics*, vol. 264, pp. 65–70, 2014, https://doi.org/10.1016/j.cam.2014.01.002.

[15] A. Bouchenak, I. M. Batiha, I. H. Jebril, M. Aljazzazi, H. Alkasasbeh, and L. Rabhi, "Generalization of the nonlinear Bernoulli conformable fractional differential equations with applications," *WSEAS Transactions on Mathematics*, vol. 24, pp. 168–180, 2025, https://doi.org/10.37394/23206.2025.24.17.

[16] A. Bouchenak, I. M. Batiha, R. Hatamleh, M. Aljazzazi, I. H. Jebril, and M. Al-Horani, "Study and analysis of the second order constant coefficients and Cauchy–Euler equations via modified conformable operator," *International Journal of Robotics and Control Systems*, vol. 5, no. 2, pp. 794–812, 2025, https://doi.org/10.31763/ijrcs.v5i2.1577.

[17] T. Abdeljawad, "On conformable fractional calculus," *Journal of Computational and Applied Mathematics*, vol. 279, pp. 57–66, 2015, https://doi.org/10.1016/j.cam.2014.10.016.

[18] W. S. Chung, "Fractional Newton mechanics with conformable fractional derivative," *Journal of Computational and Applied Mathematics*, vol. 290, pp. 150–158, 2015, https://doi.org/10.1016/j.cam.2015.04.049.

[19] M. Al Horani, M. Abu Hammad, and R. Khalil, "Variation of parameters for local fractional nonhomogenous linear differential equations," *Journal of Mathematics and Computer Science*, vol. 16, no. 2, pp. 147–153, 2016, https://doi.org/10.22436/JMCS.016.02.03.

[20] R. Khalil, M. Al Horani, and D. Anderson, "Undetermined coefficients for local fractional differential equations," *Journal of Mathematics and Computer Science*, vol. 16, no. 2, pp. 140–146, 2016, http://dx.doi.org/10.22436/jmcs.016.02.02.

[21] N. A. Khan, O. A. Razzaq, and M. Ayaz, "Some properties and applications of conformable fractional Laplace transform (CFLT)," *Journal of Fractional Calculus and Applications*, vol. 9, no. 1, pp. 72–81, 2018, https://journals.ekb.eg/article_308446_1e4877389924d044f7e8a7eea19b7f06.pdf.

[22] Z. Al-Zhour, F. Alrawajeh, N. Al-Mutairi, and R. Alkhasawneh, "New results on the conformable fractional Sumudu transforms: theories and applications," *International Journal of Analysis and Applications*, vol. 17, no. 6, pp. 1019–1033, 2019, https://doi.org/10.28924/2291-8639-17-2019-1019.

[23] M. Ayata and O. Ozkan, "A new application of conformable Laplace decomposition method for fractional Newell-Whitehead-Segel equation," *AIMS Mathematics*, vol. 5, no. 6, pp. 7402–7412, 2020, https://doi.org/10.3934/math.2020474.

[24] F. Seddiki, M. Al Horani, and R. Khalil, "Finite rank solution for conformable degenerate first-order abstract Cauchy problem in Hilbert spaces," *European Journal of Pure and Applied Mathematics*, vol. 14, no. 2, pp. 493–505, 2021, https://doi.org/10.29020/NYBG.EJPAM.V14I2.3950.

[25] F. Seddiki, M. Al Horani, and R. Khalil, "Tensor product and inverse fractional abstract Cauchy problem," *Rendiconti del Circolo Matematico di Palermo Series 2*, vol. 72, no. 4, pp. 2321–2332, 2023, https://doi.org/10.1007/s12215-022-00769-0.

[26] A. Bouchenak, I. M. Batiha, M. Aljazzazi, I. H. Jebril, M. Al-Horani, and R. Khalil, "Atomic exact solution for some fractional partial differential equations in Banach spaces," *Partial Differential Equations in Applied Mathematics*, vol. 9, p. 100626, 2024, https://doi.org/10.1016/j.padiff.2024.100626.

[27] A. Gökdoğan, E. Ünal, and E. Çelik, "Existence and uniqueness theorems for sequential linear conformable fractional differential equations," *Miskolc Mathematical Notes*, vol. 17, no. 1, pp. 267–279, 2015, https://doi.org/10.18514/MMN.2016.1635.

[28] W. Zhong and L. Wang, "Basic theory of initial value problems of conformable fractional differential equations," *Advances in Difference Equations*, vol. 2018, no. 321, pp. 1–14, 2018, https://doi.org/10.1186/s13662-018-1778-5.

[29] S. Zhang, S. Li, and L. Hu, "The existence and uniqueness result of solutions to initial value problems of nonlinear diffusion equations involving with the conformable variable derivative," *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, vol. 113, no. 2, pp. 1601–1623, 2019, https://doi.org/10.1007/s13398-018-0572-2.

[30] G. Xiao, J. Wang, and D. O'Regan, "Existence, uniqueness and continuous dependence of solutions to conformable stochastic differential equations," *Chaos, Solitons & Fractals*, vol. 139, p. 110269, https://doi.org/10.1016/j.chaos.2020.110269.

[31] P. Agarwal, M. R. Sidi Ammi, and J. Asad, "Existence and uniqueness results on time scales for fractional nonlocal thermistor problem in the conformable sense," *Advances in Difference Equations*, vol. 2021, no. 162, 2021, https://doi.org/10.1186/s13662-021-03319-7.

[32] M. Sher, A. Khan, K. Shah, and T. Abdeljawad, "Existence and stability theory of pantograph conformable fractional differential problem," *Thermal Science*, vol. 27, no. 1, pp. 237–244, 2023, https://doi.org/10.2298/TSCI23S1237S.

[33] M. Aljazzazi, S. Fakhreddine, I. M. Batiha, I. H. Jebril, A. Bouchenak, and R. I. Hajaj, "Impulsive conformable evolution equations in Banach spaces with fractional semigroup," *Filomat*, vol. 38, no. 26, pp. 9321–9332, 2024, https://www.jstor.org/stable/27387904.

[34] M. Li, J. Wang, and D. O'Regan, "Existence and Ulam's stability for conformable fractional differential equations with constant coefficients," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 42, no. 4, pp. 1791–1812, 2019, https://doi.org/10.1007/s40840-017-0576-7.

[35] S. Öğrekçi, Y. Başcı, and A. Mısır, "Ulam type stability for conformable fractional differential equations," *Rendiconti del Circolo Matematico di Palermo Series 2*, vol. 70, no. 2, pp. 807–817, 2021, https://doi.org/10.1007/s12215-020-00532-3.

[36] F. Wan, X. Liu, and M. Jia, "Ulam-Hyers stability for conformable fractional integro-differential impulsive equations with the antiperiodic boundary conditions," *AIMS Mathematics*, vol. 7, no. 4, pp. 6066–6083, 2022, http://dx.doi.org/10.3934/math.2022338.

[37] W. Albalawi, M. I. Liaqat, F. U. Din, K. S. Nisar, and A. H. Abdel-Aty, "Well-posedness and Ulam-Hyers stability results of solutions to pantograph fractional stochastic differential equations in the sense of conformable derivatives," *AIMS Mathematics*, vol. 9, no. 5, pp. 12375–12398, 2024, https://dx.doi.org/%2010.3934/math.2024605.

[38] R. P. Agarwal, M. Meehan, and D. O'Regan, *Fixed Point Theory and Applications*, Cambridge University Press, 2001, https://books.google.co.id/books?id=iNccsDsdbIcC&hl=id&source=gbs_navlinks_s.