International Journal of Robotics and Control Systems
Vol. 5, No. 2, 2025, pp. 1238-1253
ISSN 2775-2658
http://pubs2.ascee.org/index.php/ijrcs

Trapezoidal Scheme for the Numerical Solution of Fractional Initial Value Problems

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ARTICLE INFO

Article History
Received February 09, 2025
Revised April 01, 2025
Accepted May 09, 2025

Keywords
Fractional Calculus;
Fractional Initial Value Problem;
Fractional Taylor Formula;
Fractional Trapezoid Method

ABSTRACT

The purpose of this study is to recall the main concepts and definitions in relation to the fractional calculus. In light of this overview, we will propose a novel fractional version of the so-called Trapezoid method named by the fractional Trapezoid method. Such a method will then be used to numerically solve the analog version of the initial value problems called fractional initial value problem FIVPs. As consequences of the proposed numerical approach, several numerical examples will be illustrated to verify the efficiency of the proposed numerical approach.

1. Introduction

Fractional calculus is a useful tool for simulating memory and genetic characteristics in a variety of systems since it expands on standard calculus by allowing differentiation and integration to non-integer orders [1]–[3]. By extending ideas like derivatives and integrals to fractional (non-integer) orders, it offers fresh perspectives in disciplines like finance, engineering, and physics. Complex dynamical systems, viscoelastic materials, and anomalous diffusion are all explained within this mathematical paradigm. Its versatility and wide-ranging influence may be seen in its applications, which span from signal processing to control theory [4]–[10]. Overall, a richer language for explaining phenomena in the actual world is offered by fractional calculus [11].

Differential equations with fractional derivatives and initial conditions are the subject of fractional initial value problems, or FIVPs. By including non-integer order derivatives, these problems extend the scope of conventional starting value problems and allow for more precise modeling of systems with memory effects [12]–[14]. Because fractional orders contribute complexity, solving FIVPs calls for specific numerical and analytical techniques [15]. FIVPs are used in a wide range of domains, such as economics, biology, and physics, to simulate the complexities of real-world processes. It is essential to comprehend and resolve FIVPs in order to advance the use many scientific and engineering applications of fractional calculus.

One numerical method for resolving initial value problems with ordinary differential equations is the Trapezoid method. With more accuracy, it approximates the solution by splitting the region under the curve into trapezoids rather than rectangles. This approach is implicit, which improves its stability, particularly for stiff equations, as it necessitates solving a system of equations at each stage. It entails iteratively updating the answer while accounting for the function's slope at both the present and subsequent time steps. Although this approach is more computationally demanding than other, more straightforward approaches like the Euler method, it provides a more favorable balance between precision and stability, making it appropriate for a wide range of scientific and engineering applications.

2. Basic Fundamentals

This section reviews several key definitions and preliminaries related to fractional calculus. This would pave the way for the major findings later on.

Definition 2.1 Let α be a real non-negative number. Then J_a^α, defined on $L_1[a,b]$ where $L_1[a,b]$ is the set of all functions such that their absolute values are integrable on $[a,b]$, is given by [16]

$$J_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-w)^{\alpha-1} f(w)dw, \ a \leq t \leq b, \tag{1}$$

which is called the Riemann-Liouville fractional-order integral operator of order α.

In what follow, we mention some properties of the Riemann-Liouville fractional-order integral operator:

1. Let $m, n \geq 0$ and $f \in L_1[a,b]$. Then we have $J_a^m J_a^n f = J_a^{m+n} f$.
2. For $m, n \geq 0$, we have $J_a^m J_a^n f = J_a^n J_a^m f$.
3. For $m, n \geq 0$, we have $J_a^{m+n} f = J_a^{m+n-1} J_a^1 f$.

Definition 2.2 Let $\alpha \in \mathbb{R}$ and $r = \lceil \alpha \rceil$. The operator D_a^α defined by

$$D_a^\alpha f = D^r J_a^{r-\alpha} f, \tag{2}$$

is called the Riemann-Liouville fractional-order differential operator of order α.

Definition 2.3 Let $\alpha \in \mathbb{R}$ and $r = \lceil \alpha \rceil$. The Caputo fractional derivative operator D_a^α is defined by

$$D_a^\alpha f = J_a^{r-\alpha} D^r f. \tag{3}$$

Definition 2.4 Let $\alpha \in \mathbb{R}^+$ and $r = \lceil \alpha \rceil$ such that $r-1 < \alpha \leq r$. Then the Caputo fractional-order derivative operator of order α is given by [16]

$$D_a^\alpha f(t) = \frac{1}{\Gamma(r-\alpha)} \int_a^t (t-\tau)^{r-\alpha-1} f^{(r)}(\tau)d\tau, \ t > a. \tag{4}$$

In view of equation (4), the power rule property can be obtained as follows:

$$D_*^\alpha t^p = \begin{cases} \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha} & , \ r-1 < \alpha \leq r, \ p > r-1, \ p \in \mathbb{R} \\ 0 & , \ r-1 < \alpha \leq r, \ p \leq r-1, \ p \in \mathbb{N}. \end{cases} \tag{5}$$

Theorem 2.5 (**Generalized Taylor's Formula**) Suppose that $D_*^{k\alpha} f(x) \in C((0,b])$, for $k = 1, 2, \cdots, n+1$, where $0 < \alpha \leq 1$. Then, we have

$$f(x) = \sum_{i=0}^{n} \frac{(x-0)^{i\alpha}}{\Gamma(i\alpha+1)} (D_*^{i\alpha} f)(0_+) + \frac{(D_*^{(n+1)\alpha} f)(\zeta)}{\Gamma((n+1)\alpha+1)} x^{(n+1)\alpha}, \tag{6}$$

Where $\quad 0 \leq \zeta \leq x, \quad \forall\, x \in (0,b]$ [17].

3. Fractional Trapezoid Method

This part will address the main contribution of our work by illustrating the derivation of the fractional Trapezoid method. This would be carried out by deriving the fractional forward and fractional backward formulas. For this purpose, we consider the following fractional initial value problem in the Caputo sense:

$$D^{\alpha} y(t) = f(t, y(t)), \tag{7}$$

With initial condition

$$y(0) = y_0. \tag{8}$$

Here, we consider $D_*^{k\alpha} \in C(0, b]$ for $k = 0, 1, 2, \cdots, n+1$, where $0 < \alpha \leq 1$. Then, we may use the generalized Taylor method to expand f around $x = x_0$ as follows:

$$f(x) = \sum_{i=0}^{n} \frac{(x - x_0)^{i\alpha}}{\Gamma(i\alpha + 1)} D_*^{i\alpha} f(x_0) + \frac{(x - x_0)^{(n+1)\alpha}}{\Gamma((n+1)\alpha + 1)} D_*^{(n+1)\alpha} f(\xi). \tag{9}$$

Where $x \in (0, b]$ and $0 < \xi < b$. In other words, we can have

$$\begin{aligned} f(x) = &\frac{(x - x_0)^{0\alpha}}{\Gamma(0\alpha + 1)} D_*^{0\alpha} f(x_0) + \frac{(x - x_0)^{\alpha}}{\Gamma(\alpha + 1)} D_*^{\alpha} f(x_0) + \frac{(x - x_0)^{2\alpha}}{\Gamma(2\alpha + 1)} D_*^{2\alpha} f(x_0) + \cdots \\ &+ \frac{(x - x_0)^{n\alpha}}{\Gamma(n\alpha + 1)} D_*^{n\alpha} f(x_0) + \frac{(x - x_0)^{(n+1)\alpha}}{\Gamma((n+1)\alpha + 1)} D_*^{(n+1)\alpha} f(\xi). \end{aligned} \tag{10}$$

That is, we have

$$\begin{aligned} f(x) = &f(x_0) + \frac{(x - x_0)^{\alpha}}{\Gamma(\alpha + 1)} D_*^{\alpha} f(x_0) + \frac{(x - x_0)^{2\alpha}}{\Gamma(2\alpha + 1)} D_*^{2\alpha} f(x_0) + \cdots + \frac{(x - x_0)^{n\alpha}}{\Gamma(n\alpha + 1)} D_*^{n\alpha} f(x_0) \\ &+ \frac{(x - x_0)^{(n+1)\alpha}}{\Gamma((n+1)\alpha + 1)} D_*^{(n+1)\alpha} f(\xi) \end{aligned} \tag{11}$$

Where $x \in (0, b]$ and $\xi \in (0, b)$.

3.1. Forward Fractional Euler Method

To solve the problem of fractional initial value problem (7)-(8), we divide the interval $(0, b]$ as $0 = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$ for which $x_i = x_0 + ih$, $\forall i = 1, 2, 3, \cdots, n$ and $h = \frac{b}{n}$. In this regard, if the generalized Taylor formula (9) is employed to expand the function $y(x)$ around $x = x_i$, we get

$$y(x) = y(x_i) + \frac{(x - x_i)^{\alpha}}{\Gamma(\alpha + 1)} D_*^{\alpha} y(x_i) + \frac{(x - x_i)^{2\alpha}}{\Gamma(2\alpha + 1)} D_*^{2\alpha} y(\xi), \tag{12}$$

for $\xi \in (0, x)$. By substituting x_{i+1} in the equality above, we get

$$y(x_{i+1}) = y(x_i) + \frac{(x_{i+1} - x_i)^{\alpha}}{\Gamma(\alpha + 1)} D_*^{\alpha} y(x_i) + \frac{(x_{i+1} - x_i)^{2\alpha}}{\Gamma(2\alpha + 1)} D_*^{2\alpha} y(\xi), \tag{13}$$

for $\xi \in (0, x)$. In other words, we have

$$y(x_{i+1}) = y(x_i) + \frac{h^{\alpha}}{\Gamma(\alpha + 1)} D_*^{\alpha} y(x_i) + \frac{h^{2\alpha}}{\Gamma(2\alpha + 1)} D_*^{2\alpha} y(\xi), \tag{14}$$

for $\xi \in (0, x)$. Formula (14) is called the forward fractional Euler formula, which can be used to obtain an approximate solution for the FIVP (7)-(8).

3.2. Backward Fractional Euler Method

To derive the backward fractional Euler method, the generalized Taylor formula (9) is employed again to re-expand the function $y(x)$ about $x = x_{i+1}$ as follows:

$$y(x) = y(x_{i+1}) + \frac{(x - x_{i+1})^{\alpha}}{\Gamma(\alpha+1)} D_*^{\alpha} y(x_{i+1}) + \frac{(x - x_{i+1})^{2\alpha}}{\Gamma(2\alpha+1)} D_*^{2\alpha} y(\xi), \tag{15}$$

for $\xi \in (0, x)$. By substituting x_i in the equality above, we get

$$y(x_i) = y(x_{i+1}) + \frac{(-h)^{\alpha}}{\Gamma(\alpha+1)} D_*^{\alpha} y(x_i) + \frac{h^{2\alpha}}{\Gamma(2\alpha+1)} D_*^{2\alpha} y(\xi), \tag{16}$$

for $\xi \in (0, x)$. This is equivalent to say that

$$y(x_{i+1}) = y(x_i) + \frac{h^{\alpha}}{\Gamma(\alpha+1)} D_*^{\alpha} y(x_{i+1}) - \frac{h^{2\alpha}}{\Gamma(2\alpha+1)} D_*^{2\alpha} y(\xi), \tag{17}$$

for $\xi \in (0, x)$. In fact, formula (17) is called the backward fractional Euler formula, which can be also used to obtain an approximate solution for the FIVP (7)-(8).

3.3. Fractional Trapezoid method For Solving FIVP

The fractional Trapezoid procedure is a combination of the forward fractional Euler method and the backward fractional Euler method. That is, by adding (14) and (17), we get

$$2y(x_{i+1}) = 2y(x_i) + \frac{h^{\alpha}}{\Gamma(\alpha+1)} \left(D_*^{\alpha} y(x_i) + D_*^{\alpha} y(x_{i+1}) \right). \tag{18}$$

This gives

$$y(x_{i+1}) = y(x_i) + \frac{h^{\alpha}}{2\Gamma(\alpha+1)} (D_*^{\alpha} y(x_i) + D_*^{\alpha} y(x_{i+1})), \tag{19}$$

for $i = 1, 2, 3, \cdots, n$. Now, in light of the FIVP (7)-(8), we can get

$$y(x_{i+1}) = y(x_i) + \frac{h^{\alpha}}{2\Gamma(\alpha+1)} \left(f(x_i, y(x_i)) + f(x_{i+1}, y(x_{i+1})) \right), \tag{20}$$

for $i = 0, 1, 2, \cdots, n$. In fact, formula (20) is called the fractional Trapezoid formula, which can be carried out to find an approximate solution to the FIVP (7)-(8) as well.

4. Stability Analysis

The resolution of a fractional differential equation is considered steady if a tiny perturbation doesn't cause deliverance from the solution. An approach of solving a fractional differential equation numerically is stable if a small perturbation doesn't cause the numerical solution to diverge without bound. To discuss the stability analysis of the considered methods, we consider the following FIVP:

$$\begin{aligned} D^{\alpha} y(t) &= \lambda y(t), \\ y(t_0) &= y_0. \end{aligned} \tag{21}$$

It should be noted here that the solution of the FIVP (21) is stable when $\alpha = 1$ if $Re(\lambda) \leq 0$.

4.1. Stability of the Forward Fractional Euler Method

Consider we have the FIVP (21), then by implementing the forward fractional Euler technique, we arrive at

$$y_{n+1} = y_n + \frac{h^\alpha}{\Gamma(\alpha+1)} f(t_n, y_n),$$

or

$$y_{n+1} = y_n + \frac{h^\alpha}{\Gamma(\alpha+1)} (\lambda y_n),$$

or

$$y_{n+1} = \left(1 + \frac{\lambda h^\alpha}{\Gamma(\alpha+1)}\right) y_n. \tag{22}$$

This is the general explicit solution of the FIVP (21). In this regard, we can have the following states:

$$\begin{aligned}
n = 0,\ y_1 &= \left(1 + \frac{\lambda h^\alpha}{\Gamma(\alpha+1)}\right) y_0 \\
n = 1,\ y_2 &= \left(1 + \frac{\lambda h^\alpha}{\Gamma(\alpha+1)}\right) y_1 = \left(1 + \frac{\lambda h^\alpha}{\Gamma(\alpha+1)}\right)^2 y_0 \\
n = 2,\ y_3 &= \left(1 + \frac{\lambda h^\alpha}{\Gamma(\alpha+1)}\right) y_2 = \left(1 + \frac{\lambda h^\alpha}{\Gamma(\alpha+1)}\right)^3 y_0. \\
&\vdots
\end{aligned}$$

In general, we can have

$$y_{n+1} = \left(1 + \frac{\lambda h^\alpha}{\Gamma(\alpha+1)}\right)^{n+1} y_0. \tag{23}$$

So, the solution is stable when $\left|1 + \frac{\lambda h^\alpha}{\Gamma(\alpha+1)}\right| \leq 1$ because if not, $\left(\text{i.e. if } \left|1 + \frac{\lambda h^\alpha}{\Gamma(\alpha+1)}\right| > 1\right)$, then the solution goes to ∞.

4.2. Stability of the Backward fractional Euler Method

The FIVP (21), using the backward fractional Euler technique, can be expressed by

$$y_{n+1} = y_n + \frac{h^\alpha}{\Gamma(\alpha+1)} f(t_{n+1}, y_{n+1}).$$

This implies that

$$y_{n+1} = y_n + \frac{h^\alpha}{\Gamma(\alpha+1)} (\lambda y_{n+1}) = \left(1 - \frac{\lambda h^\alpha}{\Gamma(\alpha+1)}\right) y_{n+1},$$

or

$$y_{n+1} = \left(\frac{1}{1 - \frac{\lambda h^\alpha}{\Gamma(\alpha+1)}}\right) y_n.$$

This solution represents an explicit solution of the FIVP (21). In light of this solution, we can have

$$\begin{aligned}
n=0,\ y_1 &= \left(\frac{1}{1-\frac{\lambda h^\alpha}{\Gamma(\alpha+1)}}\right) y_0\\
n=1,\ y_2 &= \left(\frac{1}{1-\frac{\lambda h^\alpha}{\Gamma(\alpha+1)}}\right) y_1 = \left(\frac{1}{1-\frac{\lambda h^\alpha}{\Gamma(\alpha+1)}}\right)^2 y_0\\
n=2,\ y_3 &= \left(\frac{1}{1-\frac{\lambda h^\alpha}{\Gamma(\alpha+1)}}\right)^3 y_0.\\
&\vdots
\end{aligned}$$

Generally, we can obtain

$$y_{n+1} = \left(\frac{1}{1-\frac{\lambda h^\alpha}{\Gamma(\alpha+1)}}\right)^{n+1} y_0.$$

Hence, the solution is stable if

$$\left|\frac{1}{1-\frac{\lambda h^\alpha}{\Gamma(\alpha+1)}}\right| \leq 1,$$

or equivalently if

$$\left|\frac{1}{1-\frac{\lambda h^\alpha}{\Gamma(\alpha+1)}}\right| \geq 1,$$

provided that $\frac{\lambda h^\alpha}{\Gamma(\alpha+1)} \neq 1$.

4.3. Stability of the Fractional Trapezoid Method

Herein, we also consider the FIVP (21) and the fractional Trapezoid formula, which can be re-expressed as

$$y_{n+1} = y_n + \frac{h^\alpha}{2\Gamma(\alpha+1)}\left(f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right),$$

or

$$y_{n+1} = y_n + \frac{h^\alpha}{2\Gamma(\alpha+1)}\left(\lambda y_n + \lambda y_{n+1}\right),$$

or

$$\left(1-\frac{\lambda h^\alpha}{2\Gamma(\alpha+1)}\right) y_{n+1} = \left(1+\frac{\lambda h^\alpha}{2\Gamma(\alpha+1)}\right) y_n,$$

which implies that

$$y_{n+1} = \left(\frac{1+\frac{\lambda h^\alpha}{2\Gamma(\alpha+1)}}{1-\frac{\lambda h^\alpha}{2\Gamma(\alpha+1)}}\right) y_n.$$

Consequently, the following states are obtained:

$$
\begin{aligned}
n=0,\;\; y_1 &= \left(\frac{1+\frac{\lambda h^\alpha}{2\Gamma(\alpha+1)}}{1-\frac{\lambda h^\alpha}{2\Gamma(\alpha+1)}}\right) y_0\\
n=1,\;\; y_2 &= \left(\frac{1+\frac{\lambda h^\alpha}{2\Gamma(\alpha+1)}}{1-\frac{\lambda h^\alpha}{2\Gamma(\alpha+1)}}\right) y_1 = \left(\frac{1+\frac{\lambda h^\alpha}{2\Gamma(\alpha+1)}}{1-\frac{\lambda h^\alpha}{2\Gamma(\alpha+1)}}\right)^2 y_0\\
n=2,\;\; y_3 &= \left(\frac{1+\frac{\lambda h^\alpha}{2\Gamma(\alpha+1)}}{1-\frac{\lambda h^\alpha}{2\Gamma(\alpha+1)}}\right)^3 y_0.\\
&\vdots
\end{aligned}
$$

In generally, we can have

$$
y_{n+1} = \left(\frac{1+\frac{\lambda h^\alpha}{2\Gamma(\alpha+1)}}{1-\frac{\lambda h^\alpha}{2\Gamma(\alpha+1)}}\right)^{n+1} y_0.
$$

Herein, one can observe that the above solution will be stable if

$$
\left|\frac{1+\frac{\lambda h^\alpha}{2\Gamma(\alpha+1)}}{1-\frac{\lambda h^\alpha}{2\Gamma(\alpha+1)}}\right| \le 1,
$$

Provided that $\frac{\lambda h^\alpha}{2\Gamma(\alpha+1)} \neq 1$.

5. Illustrative Examples

In this part, we shall study three numerical examples, the first one is linear while the others are nonlinear.

Example 5.1 Consider the fractional initial value problem that follows [18]:

$$
D_*^\alpha y(t) = y(t) - t^2 + 1,\; 0 \le t \le 2, \tag{24}
$$

Under the initial condition

$$
y(0) = 0.5. \tag{25}
$$

It is important to know that the exact resolution of the problem (24)-(25) is

$$
y(t) = (t+1)^2 - \frac{1}{2}e^t. \tag{26}
$$

Now, for the purpose of the obtaining an approximate solution to the problem (24)-(25) with the use of fractional Trapezoid method, we should consider the following approximate formula:

$$
\begin{aligned}
z_0 &= 0.5,\\
z_{i+1} &= z_i + \frac{h^\alpha}{\Gamma(2\alpha+1)}(f(t_i, z_i) + f(t_{i+1}, z_{i+1})).
\end{aligned} \tag{27}
$$

To apply on (27), we should first assume $h = 0.2$ and obtain consequently the term of the mesh-point t_i, which would be as

$$
t_i = a + ih = 0.2i,\; i = 0, 1, 2, \cdots, n.
$$

Thus, by applying on formula (27) with considering the above observation, we can plot Fig. 1 and Fig. 2 shown below. This figure contains a numerical comparison between the exact solution and the numerical solutions performed by the fractional Euler method and the fractional Trapezoid approach.

To increase the numerical solution's accuracy, a smaller step size of $h = 0.02$ is chosen. The resulting approximate solution is plotted alongside the exact solution to provide a detailed comparison. Fig. 3 demonstrates the close agreement between the numerical approximation using the fractional Trapezoid method and the exact solution, highlighting the precision of the method when smaller step sizes are employed shown in Table 1.

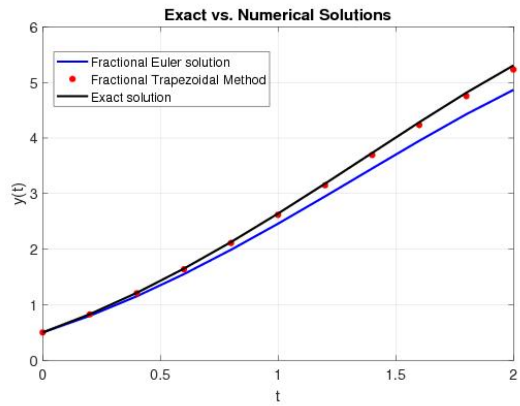

Fig. 1. Comparing numerical and exact solution for the FIVP (24)-(25) using the fractional Trapezoid method with α = 1 and $h = 0.2$

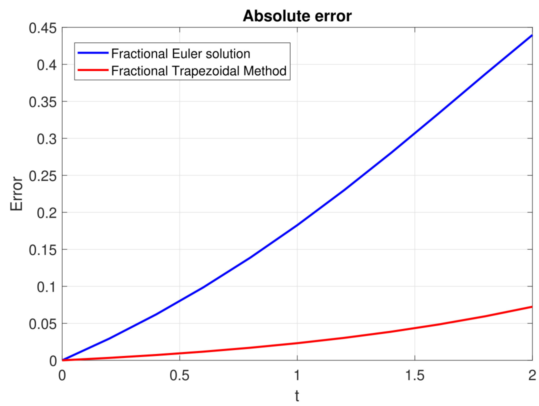

Fig. 2. The absolute error between the numerical and exact solutions of the FIVP using the fractional Trapezoid method with $\alpha = 1$ and $h = 0.2$

To visualize the accuracy of the fractional Trapezoid method, the total discrepancy between the numerical approximation and the precise solution is calculated and displayed in Fig. 4. This plot highlights the behavior of the error over time, showcasing how the numerical solution closely tracks the exact solution with minimal deviation.

The figure provides valuable insights into the effectiveness and precision of the fractional Trapezoid method when solving fractional initial value problems. To explore the impact of varying the parameter α, the approximate solutions for the FIVP (24)-(25) are plotted for different α values. Fig. 5 illustrates how the numerical solutions change as α varies, demonstrating the sensitivity of the fractional Trapezoid method to this parameter. By comparing these solutions with the exact solution, the effect of α on the accuracy and behavior of the method is highlighted.

Table 1. Absolute errors between the exact and the fractional Euler and Trapezoid solutions of problem (24)-(25) for $\alpha = 1$ and $h = 0.02$

t	**Fractional Euler method**	**Fractional Trapezoid method**
0	0	0
0.2	0.029298620919915	0.003298620919915
0.4	0.062087651179364	0.007167651179364
0.6	0.098540599804746	0.011698199804746
0.8	0.138749535753766	0.016993807753766
1.0	0.182683085770477	0.023171497610476
1.2	0.230130338631727	0.030362681076526
1.4	0.280626576577662	0.038713810360317
1.6	0.333355659802442	0.048386616217281
1.8	0.387022514193524	0.059557718459628
2.0	0.439687446214673	0.072417320347319

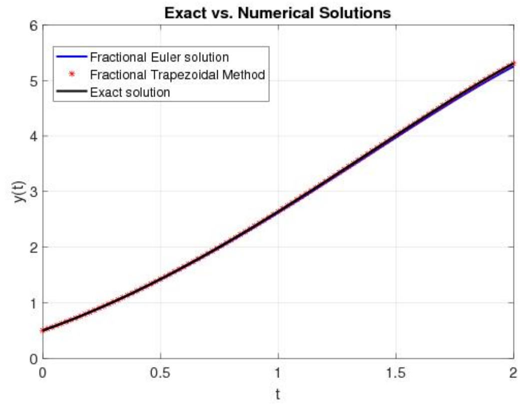

Fig. 3. Comparison of the numerical and exact solutions of the FIVP (24)-(25) using the fractional Trapezoid method with $\alpha = 1$ and $h = 0.02$

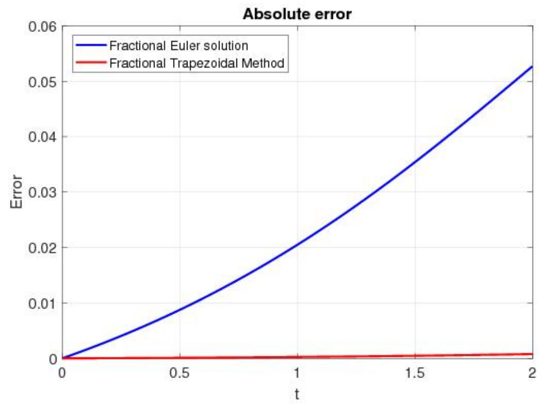

Fig. 4. The absolute error between the numerical and exact solutions for the FIVP (24)-(25) using the fractional Trapezoid method with $\alpha = 1$ and $h = 0.02$

Example 5.2 Consider the fractional initial value problem that follows [18]:

$$D_*^{\alpha} y(t) = e^{t-y}, \tag{28}$$

with the initial condition

$$y(0) = 1. \tag{29}$$

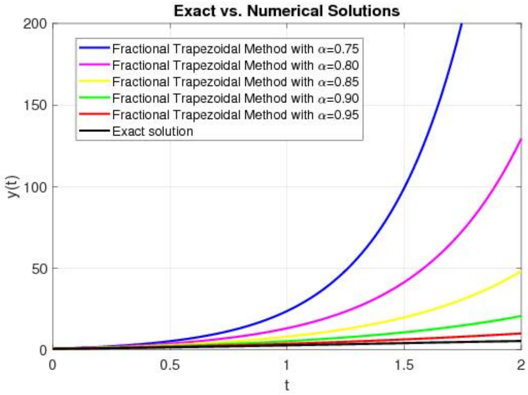

Fig. 5. Comparison of approximate solutions using the fractional Trapezoid method for different fractional values and the exact solution when $h = 0.02$

It should be noted that the precise solution to the aforementioned problem is provided by

$$y(t) = log(e^t + e^1 - 1). \tag{30}$$

Now, for the purpose of the obtaining an approximate solution to the problem (28)-(29) with the use of fractional Trapezoid method, we should consider the following approximate formula:

$$\begin{aligned} z_0 &= 0.5. \\ z_{i+1} &= z_i + \frac{h^\alpha}{\Gamma(2\alpha+1)}(f(t_i, z_i) + f(t_{i+1}, z_{i+1})). \end{aligned} \tag{31}$$

By using MATLAB, the numerical solution of the FIVP described in equations (31) is visualized through a graphical representation as shown in Fig. 6. The plot illustrates the comparison between the numerical solution obtained through the fractional Trapezoid method and the exact analytical solution. This comparison highlights the method's precision and demonstrates its effectiveness in accurately approximating solutions.

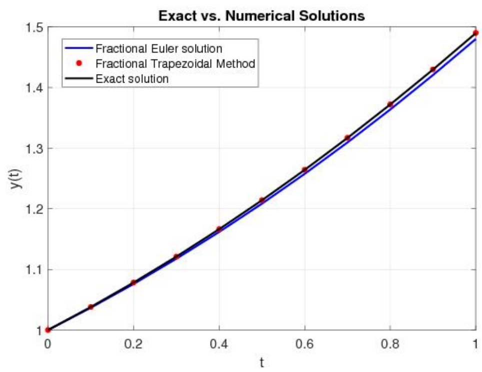

Fig. 6. Comparison of numerical and exact solutions for the FIVP (28)-(29) using the fractional Trapezoid method with $\alpha = 1$ and $h = 0.1$

To evaluate the precision of the fractional Trapezoid method, the total discrepancy between the numerical approximation and the exact solution for the FIVP (28)-(29) is calculated and depicted in Fig. 7 and Table 2. A plot of the absolute error over time is presented, showing how the error gradually increases as t grows. This visualization provides a clear understanding of the method's accuracy and effectiveness in approximating the solution.

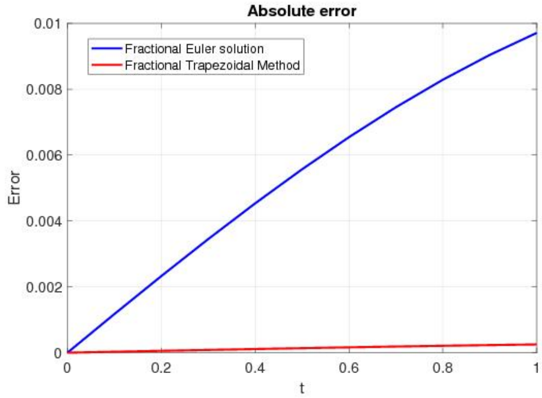

Fig. 7. Plot of the absolute error between the numerical and exact solutions of the FIVP (28)-(29) using the fractional Trapezoid method with $\alpha = 1$ and $h = 0.1$

Table 2. Absolute errors between the exact and the fractional Euler and Trapezoid solutions of problem (28)-(29) for $\alpha = 1$ and $h = 0.1$

t	**Fractional Euler method**	**Fractional Trapezoid method**
0	0	0
0.1	0.001172570096938	0.027686397851889
0.2	0.002325890908587	0.055392752877470
0.3	0.003449801503171	0.082890106308797
0.4	0.004534173462960	0.109938585495684
0.5	0.005569170453683	0.136292824878925
0.6	0.006545510426157	0.161707987452964
0.7	0.007454717546495	0.185946123326008
0.8	0.008289350989101	0.208782561090137
0.9	0.009043198812865	0.230012009821534
1.0	0.009711427206174	0.249454057768217

In order to increase the numerical solution's accuracy, a smaller size of steps of $h = 0.01$ is chosen. The resulting approximate solution is plotted alongside the exact solution to provide a detailed comparison. Fig. 8 demonstrates the close agreement between the numerical approximation using the fractional Trapezoid method and the exact solution, highlighting the precision of the method when smaller step sizes are employed.

To visualize the accuracy of the fractional Trapezoid method, the absolute discrepancy between the exact solution and the numerical approximation is calculated and displayed in Fig. 9. This plot highlights the behavior of the error over time, showcasing how the numerical solution closely tracks the exact solution with minimal deviation. The figure provides valuable insights into the effectiveness and precision of the fractional Trapezoid method when solving fractional initial value problems.

To explore the impact of varying the parameter α, the approximate solutions for the FIVP (28)-(29) are plotted for different fractional values. Fig. 10 illustrates how the numerical solutions change as α varies, demonstrating the sensitivity of the fractional Trapezoid method to this parameter. By comparing these solutions with the exact solution, the effect of α on the accuracy and behavior of the method is highlighted.

Example 5.3 Consider the following fractional initial value problem [18]:

$$D_*^{\alpha} y(t) = (y(t))^2, \tag{32}$$

with the initial condition

$$y(0) = 1. \tag{33}$$

It should be noted that the precise solution to the aforementioned problem is provided by

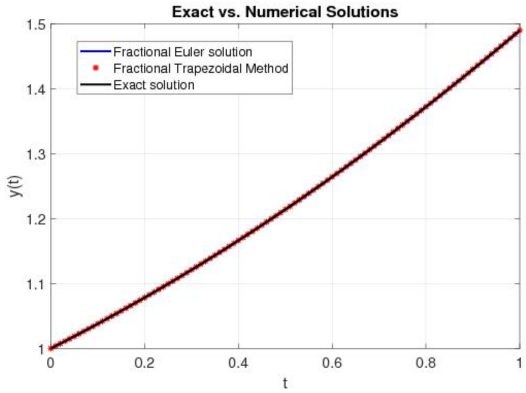

Fig. 8. Comparison of the numerical and exact solutions of the FIVP (28)-(29) using the fractional Trapezoid method with $\alpha = 1$ and $h = 0.01$

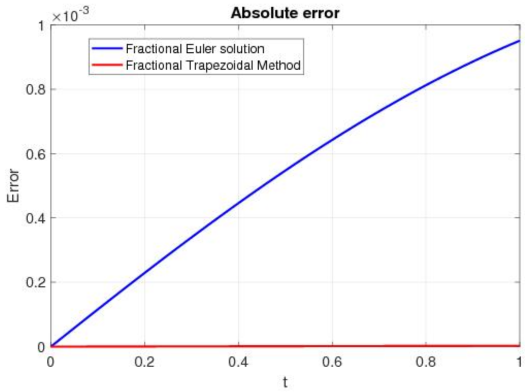

Fig. 9. Plot of the absolute error between the numerical and exact solutions for the FIVP (28)-(29) using the fractional Trapezoid method with $\alpha = 1$ and $h = 0.01$

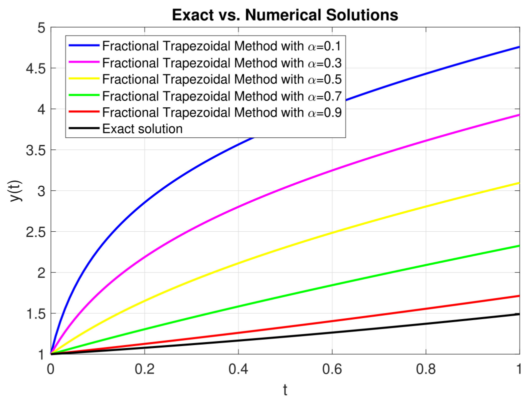

Fig. 10. Comparison of approximate solutions using the fractional Trapzoid method for different α values and the exact solution and $h = 0.01$

$$y(t) = \frac{1}{1-t}. \tag{34}$$

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Now, for the purpose of the obtaining an approximate solution to the problem (32)-(33) with the use of fractional Trapezoid method, we should consider the following approximate formula:

$$
\begin{aligned}
z_0 &= 0.5. \\
z_{i+1} &= z_i + \frac{h^\alpha}{\Gamma(2\alpha+1)}(f(t_i, z_i) + f(t_{i+1}, z_{i+1})).
\end{aligned} \tag{35}
$$

Now, by using MATLAB, the numerical solution of the FIVP (32)-(33) described in equations (35) is visualized through a graphical representation as shown in Fig. 11. The plot illustrates the comparison between the numerical solution obtained through the fractional Trapezoid method and the exact analytical solution. This comparison highlights the method's precision and demonstrates its effectiveness in accurately approximating solutions.

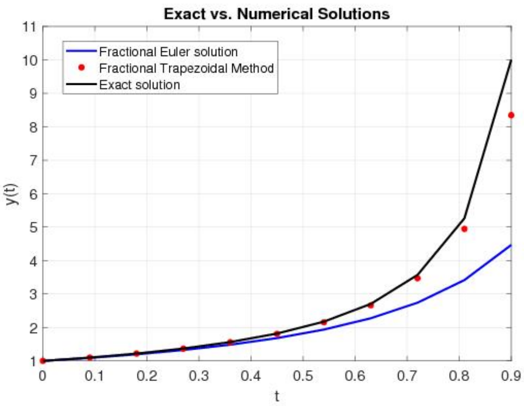

Fig. 11. Comparison of numerical and exact solutions for the FIVP (32)-(33) using the fractional Trapezoid method with $\alpha = 1$ and $h = 0.09$

To evaluate the precision of the fractional Trapezoid method, the absolute discrepancy between the exact solution and the numerical approximation for the FIVP is calculated and depicted in Fig. 12 and Table 3. A plot of the absolute error over time is presented, showing how the error gradually increases as t grows. This visualization provides a clear understanding of the method's accuracy and effectiveness in approximating the solution.

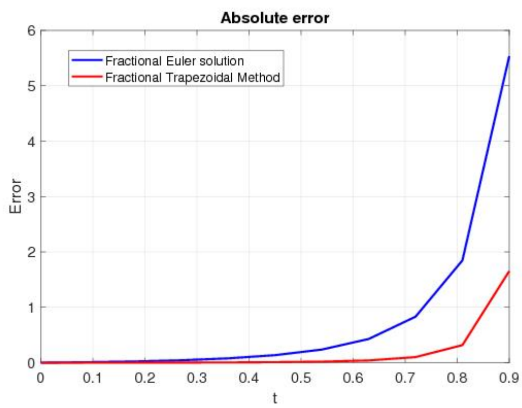

Fig. 12. Plot of the absolute error between the numerical and exact solutions of the FIVP (32)-(33) using the fractional Trapezoid method with $\alpha = 1$ and $h = 0.09$

In order to increase the numerical solution's accuracy, a smaller step size of $h = 0.009$ is chosen. The resulting approximate solution is plotted alongside the exact solution to provide a detailed comparison. Fig. 13 demonstrates the close agreement between the numerical approximation using the fractional Trapezoid method and the exact solution, highlighting the precision of the method when smaller step sizes are employed.

Table 3. Absolute errors between the exact and the fractional Euler and Trapezoid solutions of problem (32)-(33) for $\alpha = 1$ and $h = 0.09$

t	**Fractional Euler method**	**Fractional Trapezoid method**
0	0	0
0.09	0.008901098901099	0.000436598901099
0.18	0.022583195121951	0.001184806341185
0.27	0.043996500904940	0.002495651761202
0.36	0.078420506329017	0.004877901816391
0.45	0.135878049592710	0.009446776806998
0.54	0.236896137606333	0.018909519376535
0.63	0.428002692403745	0.040810174187333
0.72	0.831045148812746	0.100748714244161
0.81	1.846901354855732	0.316851989153695
0.90	5.533370672953649	1.653346172476256

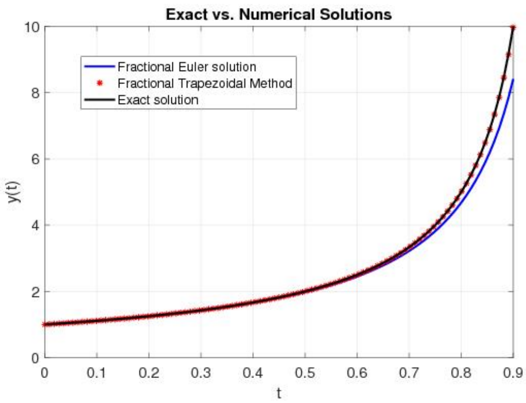

Fig. 13. Comparison of the numerical and exact solutions of the FIVP (32)-(33) using the fractional Trapezoid method with $\alpha = 1$ and $h = 0.009$

To visualize the accuracy of the fractional Trapezoid method, the absolute error between the numerical approximation and the exact solution is calculated and displayed in Fig. 14. This plot highlights the behavior of the error over time, showcasing how the numerical solution closely tracks the exact solution with minimal deviation. The figure provides valuable insights into the effectiveness and precision of the fractional Trapezoid method when solving α initial value problems.

To explore the impact of varying the parameter α, the approximate solutions for the FIVP (32)-(33) are plotted for different α values. Fig. 15 illustrates how the numerical solutions change as α varies, demonstrating the sensitivity of the fractional Trapezoid method to this parameter. By comparing these solutions with the exact solution, the effect of α on the accuracy and behavior of the method is highlighted.

6. Conclusion and Future Works

The purpose of this study was to recall the math concepts and definitions in relation to fractional calculus. In light of this overview, we proposed a fractional version of the so-called Trapezoid method. Such method was then used to numerically solve an analog version of the initial value problem called fractional initial value problems (FIVP).

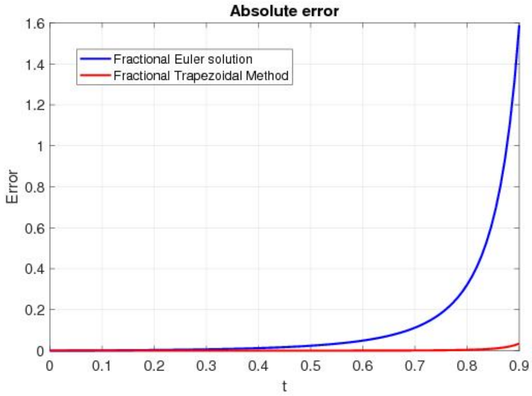

Fig. 14. Plot of the absolute error between the numerical and exact solutions for the FIVP (32)-(33) using the fractional Trapezoid method with α and $h = 0.009$

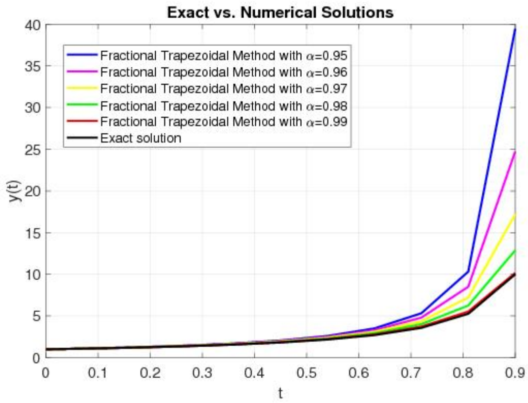

Fig. 15. Comparison of approximate solutions using the fractional Trapezoid method for different α values and the exact solution and $h = 0.009$

As a consequence, several numerical compositions were solved to verify the effectiveness of the proposed numerical approach. The method demonstrated good accuracy compared to exact solutions, particularly with small step sizes. Future work will focus on extending this scheme to nonlinear systems and partial fractional differential equations.

Author Contribution: All authors contributed equally to the main contributor to this paper. All authors read and approved the final paper.

Funding: No funding was provided.

Conflicts of Interest: The authors declare no conflict of interest.

References

[1] G. Farraj, B. Maayah, R. Khalil, and W. Beghami, "An algorithm for solving fractional differential equations using conformable optimized decomposition method," *International Journal of Advances in Soft Computing and Its Applications*, vol. 15, no. 1, 2023, https://doi.org/10.15849/IJASCA.230320.13.

[2] M. Berir, "Analysis of the effect of white noise on the Halvorsen system of variable-order fractional derivatives using a novel numerical method," *International Journal of Advances in Soft Computing and its Applications*, vol. 16, no. 3, pp. 294–306, 2024, https://doi.org/10.15849/IJASCA.241130.16.

[3] N. R. Anakira, A. Almalki, D. Katatbeh, G. B. Hani, A. F. Jameel, K. S. Al Kalbani, and M. Abu-Dawas, "An algorithm for solving linear and non-linear Volterra Integro-differential equations," *International*

Journal of Advances in Soft Computing & Its Applications, vol. 15, no. 3, pp. 77–83, 2023, https://doi.org/10.15849/IJASCA.231130.05.

[4] A. Bouchenak, I. M. Batiha, I. H. Jebril, M. Aljazzazi, H. Alkasasbeh, and L. Rabhi, "Generalization of the nonlinear Bernoulli conformable fractional differential equations with applications," *WSEAS Transactions on Mathematics*, vol. 24, pp. 168–180, 2025, https://doi.org/10.37394/23206.2025.24.17.

[5] A. Bouchenak, I. M. Batiha, R. Hatamleh, M. Aljazzazi, I. H. Jebril, and M. Al-Horani, "Study and analysis of the second order constant coefficients and Cauchy-Euler equations via modified conformable operator," *International Journal of Robotics and Control Systems*, vol. 5, no. 2, pp. 794–812, 2025, https://doi.org/10.31763/ijrcs.v5i2.1577.

[6] H. Qawaqneh, H. A. Jari, A. Altalbe, and A. Bekir, "Stability analysis, modulation instability, and the analytical wave solitons to the fractional Boussinesq–Burgers system," *Physica Scripta*, vol. 99, no. 12, p. 125235, 2024, https://doi.org/10.1088/1402-4896/ad8e07.

[7] A. Boudjedour, I. Batiha, S. Boucetta, M. Dalah, K. Zennir, and A. Ouannas, "A finite difference method on uniform meshes for solving the time-space fractional advection-diffusion equation," *Gulf Journal of Mathematics*, vol. 19, no. 1, pp. 156–168, 2025, https://doi.org/10.56947/gjom.v19i1.2524.

[8] N. Allouch, I. M. Batiha, I. H. Jebril, S. Hamani, A. Al-Khateeb, and S. Momani, "A new fractional approach for the higher-order q-Taylor method," *Image Analysis and Stereology*, vol. 43, no. 3, pp. 249–257, 2024, https://doi.org/10.5566/ias.3286.

[9] A. Zraiqat, I. M. Batiha, and S. Alshorm, "Numerical comparisons between some recent modifications of fractional Euler methods," *WSEAS Transactions on Mathematics*, vol. 23, no. 1, pp. 529–535, 2024, https://doi.org/10.37394/23206.2024.23.55.

[10] S. Momani, N. Djenina, A. Ouannas, and I. M. Batiha, "Stability results for nonlinear fractional differential equations with incommensurate orders," *IFAC-PapersOnLine*, vol. 58, no. 12, pp. 286–290, 2024, https://doi.org/10.1016/j.ifacol.2024.08.204.

[11] C. B. Boyer and U. C. Merzbach, *A History of Mathematics*, John Wiley & Sons, 2011, https://books.google.co.id/books?id=V6RUDwAAQBAJ&hl=id&source=gbs_navlinks_s

[12] I. M. Batiha, I. H. Jebril, N. Anakira, A. A. Al-Nana, R. Batyha, and S. Momani, "Two-dimensional fractional wave equation via a new numerical approach," *International Journal of Innovative Computing, Information & Control*, vol. 20, no. 4, pp. 1045–1059, 2024, https://doi.org/10.24507/ijicic.20.04.1045.

[13] S. Momani, M. Shqair, I. M. Batiha, M. H. E. Abu-Sei'leek, S. Alshorm, and S. A. Abd El-Azeem, "Two-energy group neutron diffusion model in spherical reactors," *Results in Nonlinear Analysis*, vol. 7, no. 2, pp. 160–173, 2024, https://doi.org/10.31838/rna/2024.07.02.013.

[14] S. Momani, I. M. Batiha, A. Abdelnebi, and I. H. Jebril, "A powerful tool for dealing with high-dimensional fractional-order systems with applications to fractional Emden–Fowler systems," *Chaos, Solitons & Fractals: X*, vol. 12, p. 100110, 2024, https://doi.org/10.1016/j.csfx.2024.100110.

[15] K. Diethelm and N. J. Ford, "The Analysis of Fractional Differential Equations," *Journal of Mathematical Analysis and Applications*, vol. 265, no. 2, pp. 229–248, 2002, https://doi.org/10.1006/jmaa.2000.7194.

[16] I. Podlubny, *Fractional Differential Equations*, Elsevier, 1998, https://books.google.co.id/books?id=K5FdXohLto0C&hl=id&source=gbs_navlinks_s.

[17] Z. M. Odibat and S. Momani, "An algorithm for the numerical solution of differential equations of fractional order," *Journal of Applied Mathematics and Informatics*, vol. 26, no. 1-2, pp. 15–27, 2008, https://koreascience.kr/article/JAKO200833338752380.pdf.

[18] W. Gauts, *Numerical Analysis*, Springer Science & Business Media, 2011, https://books.google.co.id/books?id=-fgjJF9yAIwC&hl=id&source=gbs_navlinks_s.