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Study and Analysis of the Second Order Constant Coefficients and Cauchy-Euler Equations via Modified Conformable Operator

Ahmed Bouchenak [a,b,1], Iqbal M. Batiha [c,d,2,*], Raed Hatamleh [e,3], Mazin Aljazzazi [f,4], Iqbal H. Jebril [c,5], Mohammed Al-Horani [f,6]

[a] Department of Mathematics, Faculty of Exact Sciences, University Mustapha Stambouli of Mascara, Mascara, 29000, Algeria
[b] Mathematics Research Center, Near East University, Nicosia, 99138, Turkey
[c] Department of Mathematics, Al Zaytoonah University of Jordan, Amman 11733, Jordan
[d] Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman 346, United Arab Emirates
[e] Department of Mathematics, Faculty of Science, Jadara University, Irbid 21110, Jordan
[f] Department of Mathematics, The University of Jordan, Amman, Jordan
[1] a.bouchenak@univ-mascara.dz; [2] i.batiha@zuj.edu.jo; [3] raed@jadara.edu.jo; [4] m.jazazi@yahoo.com; [5] i.jebril@zuj.edu.jo; [6] horani@ju.edu.jo
* Corresponding Author

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ABSTRACT

In this paper, we are concerned with a new modified conformable operator. Such an operator makes the study very easy in fractional calculus because it satisfies the most properties as the usual derivative and gives exact solutions. Furthermore, we will analyze and study the second-order fractional linear homogeneous differential equation with constant coefficients, which has two reasons for the importance of these types of differential equations. First of all, they often arise in applications. Second, it is relatively easy to find fundamental sets of solutions to these equations. In addition, we will also analyze the related fractional Cauchy–Euler type equation, which is used in various fields, physics, engineering, etc. Finally, as an application, we will illustrate the method on some numerical examples of the mentioned type of fractional differential equations.

1. Introduction

Fractional derivatives emanations dates back to the times of calculus. In 1695 L‘Hopital wondered at the meaning of $\frac{d^n f}{dt^n}$ at $n = \frac{1}{2}$. So, from that time, the mathematical researchers have been attempting to define a fractional derivative. The most known are Riemann Liouville definition and Caputo definition [1]. As per Riemann Liouville definition 1847, fractional derivative of constant function is not zero. In 1967 Caputo noticed it and gave another simplified definition of fractional operator on the basis of series expansion. The theory of fractional derivatives progressed for three centuries as primarily a theoretical study of mathematics relevant only to mathematicians [2].

In the 1980’s, Mandelbrot works on fractional geometry drew the attention of physicists to this field of study and this led to the beginning of several publications in the field of fractional Brownian motion and anomalous diffusion processes, see [3] and for more see [4]–[21]. The previous definitions give an approximate solution to the problems as boundary values problems and fractional differential equations [22], [23]. The singular property that these fractional calculi have in common may be the linearity however not all of these fractional derivatives conform the classical properties like the product rule, the chain rule, etc. In order to overcome these or other poverties, in [24] the authors introduced a new simple and well behaved local derivative called conformable derivative, which defined as

Definition 1.1 [24] Given a function $f : [0, \infty) \to \mathbb{R}$. Then

$$f^{(\alpha)}(x) = \lim_{\epsilon \to 0} \frac{f(x + \epsilon x^{1-\alpha}) - f(x)}{\epsilon},$$

for all $x > 0$ and $\alpha \in (0, 1)$. If the limit exists, we call $f^{(\alpha)}(x)$ the conformable derivative of f of order α. If f is α-differentiable in some $(0, a)$ such that $a > 0$ and $\lim_{x \to 0^+} f^{(\alpha)}(x)$ exists, then define

$$f^{(\alpha)}(0) = \lim_{x \to 0^+} f^{(\alpha)}(x).$$

The conformable derivative defined on a basic limit definition and satisfies most of the properties that the classical integer order derivative has. In 2015, T. Abdeljawad generalized many beneficial and worthy results [25]. Moreover, authors developed this derivative more specifically conformable Laplace transform in [26]–[28] and conformable Fourier series in [29]–[31] and the atomic exact solution using tensor product theory in [32]. Khalil et al. in [33] presented a geometric meaning of the conformable derivative using the idea of fractional cords so from that time, more and more attention has been paid to this derivative and many problems were solved by using such a definition. More applications on conformable operator can be found in [34]–[37], and others can be found in [38]–[54].

In this paper we will be concerned with a newly defined conformable operator [55], which was introduced by Anderson, Douglas R., and Darin J. Ulness such a modified conformable operator has a great rule because it satisfy the most properties as the usual derivative. Also, this article aims to use the modified conformable operator in the study and analysis of the following type of a constant coefficients modified conformable differential equation

$$aD^\alpha D^\alpha u(x) + bD^\alpha u(x) + cu(x) = 0, \quad a, b, c \in \mathbb{R}, \quad x \in [x_0, \infty), \quad x_0 > 0,$$

this type of equations has a vital role in differential equations because of two reasons. First of all, they often arise in applications. Secondly, as we will see, it is relatively easy to find fundamental sets of solutions for these equations. Furthermore, we will study and analyze the given homogeneous Cauchy-Euler modified conformable differential equation via modified conformable operator

$$axD^\alpha \left[xD^\alpha u(x)\right] + bD^\alpha u(x) + cu(x) = 0, \quad a, b, c \in \mathbb{R}, \quad x \in [x_0, \infty), \quad x_0 > 0,$$

by presenting the general solutions of this Cauchy-Euler equations using the roots of its associated characteristic equations. For applications of the Cauchy-Euler equation in physics and engineering we refer the reader to [56]–[58]. Moreover, we should mention here that the value of the used modified conformable operator is to give an exact solution to this type equations, however authors in [59], [60] find an approximate solutions. Finally, as an application we illustrate the method of solution on some numerical example in details.

2. Modified Conformable Calculus

In this part we present some preliminaries of modified conformable operator denoted D^α of order α where $0 < \alpha \leq 1$ and D^0, D^1 are reduced to the identity operator and the classical differential operator, respectively.

Definition 2.1 (Modified Conformable Differential Operator). [55]
Let $0 < \alpha \leq 1$, a differential operator D^{α} is modified conformable iff D^0 is the identity operator and D^1 is the classical differential operator. Specifically, D^{α} is modified conformable iff for a differentiable function $f(x)$ we have

$$D^0 f(x) = f(x) \ \ and \ \ D^1 f(x) = \frac{d}{dx} f = f^{'}(x), \ \ x \in \mathbb{R} \tag{1}$$

Definition 2.1 is more general. To clarify this definition for readers, we say that the modified conformable operator D^{α} can take several forms, as determined by the following definition:

Definition 2.2 (A Class of Modified Conformable Derivative). [55]
Let $0 < \alpha \leq 1$, and let the functions $k_0, k_1 : [0, 1] \times \mathbb{R} \to [0, \infty)$ be continuous such that

$$\begin{array}{ll} \lim\limits_{\alpha \to 0^+} k_1(\alpha, x) = 1, & \lim\limits_{\alpha \to 0^+} k_0(\alpha, x) = 0, \ \ \forall x \in \mathbb{R}, \\ \lim\limits_{\alpha \to 1^-} k_1(\alpha, x) = 0, & \lim\limits_{\alpha \to 1^-} k_0(\alpha, x) = 1, \ \ \forall x \in \mathbb{R}, \\ k_1(\alpha, x) \neq 0, \ \alpha \in [0, 1), & k_0(\alpha, x) \neq 0, \ \alpha \in (0, 1], \ \ \forall x \in \mathbb{R}. \end{array} \tag{2}$$

Then the following differential operator D^{α} defined by

$$D^{\alpha} f(x) = k_1(\alpha, x) f(x) + k_0(\alpha, x) f^{'}(x) \tag{3}$$

is modified conformable operator provided that the function $f(x)$ is differentiable and $f^{'}(x) = \frac{d}{dx} f$.

The following example explains how an operator can be a class of modified conformable derivative.

Example 2.3
1. Take $k_1(\alpha, x) = (1 - \alpha)x^{\alpha}$ and $k_0(\alpha, x) = \alpha x^{1-\alpha}$ for any $x \in (0, \infty)$, we find

$$\begin{aligned} D^{\alpha} f(x) &= k_1(\alpha, x) f(x) + k_0(\alpha, x) f^{'}(x) \\ &= (1 - \alpha)x^{\alpha} f(x) + \alpha x^{1-\alpha} f^{'}(x). \end{aligned}$$

   Based on the obtained operator, we get:

$$D^0 f(x) = f(x) \ \ and \ \ D^0 f(x) = f^{'}(x), \ \ itmeans \ D^{\alpha} \ satisfy \, condition \ (2.1),$$

   and one can easily prove that D^{α} satisfy condition (2.2), then we say that D^{α} is a class of modified conformable derivative.
2. Take $k_1 = \cos\left(\frac{\alpha\pi}{2}\right) x^{\alpha}$ and $k_0 = \sin\left(\frac{\alpha\pi}{2}\right) x^{1-\alpha}$ for any $x \in (0, \infty)$ we get

$$\begin{aligned} D^{\alpha} f(x) &= k_1(\alpha, x) f(x) + k_0(\alpha, x) f^{'}(x) \\ &= \cos\left(\frac{\alpha\pi}{2}\right) x^{\alpha} f(x) + \sin\left(\frac{\alpha\pi}{2}\right) x^{1-\alpha} f^{'}(x). \end{aligned}$$

   Similarly, the resulting operator satisfy conditions (2.1) and (2.2), then is a class of modified conformable derivative. Note that unfortunately $D^{\alpha} D^{\beta} \neq D^{\beta} D^{\alpha}$ in general.

Definition 2.4 (Partial Conformable Derivatives). [55]
Let $0 < \alpha \leq 1$, and let the functions $k_0, k_1 : [0, 1] \times \mathbb{R} \to [0, \infty)$ be continuous and satisfy (2). Given a function $f(x, s) : \mathbb{R}^2 \to \mathbb{R}$ such that $\frac{d}{dx} f(x, s)$ exists for each fixed $s \in \mathbb{R}$, define the partial modified conformable differential operator D_x^{α} by

$$D_x^{\alpha} f(x, s) = k_1(\alpha, x) f(x, s) + k_0(\alpha, x) \frac{\partial}{\partial x} f(x, s) \tag{4}$$

it means that, D_x^{α} correspond to derive the function $f(x, s)$ of the two variables x and s with respect to x.

Definition 2.5 (Modified Conformable Exponential Function). [55]
Let $0 < \alpha \leq 1$, $s, x \in \mathbb{R}$ with $s \leq x$, and let the functions $m : [s, x] \to \mathbb{R}$ be continuous. $k_0, k_1 : [0, 1] \times \mathbb{R} \to [0, \infty)$ be continuous and satisfy (2) with m/k_0 and k_1/k_0 Riemann integrable on $[s, x]$. Then the modified conformable exponential function with respect to D^α is defined to be

$$e_m(x, s) = e^{\int_s^x \frac{m(\lambda)-k_1(\alpha,\lambda)}{k_0(\alpha,\lambda)} d\lambda}, \quad if\ m = 0\ then\ \ e_0(x, s) = e^{\int_x^s \frac{k_1(\alpha,\lambda)}{k_0(\alpha,\lambda)} d\lambda}. \tag{5}$$

We should mention here that the exponential function will help us to simplify the form of the solutions for the fractional differential equations that we present in the next section, also the value of the function m will be change depend to the situation.

Now based on (3) and (5), we have the following basic results.

Lemma 2.6 (Basic Derivatives). [55]
Let the modified conformable differential operator D^α be given as (3), where $0 < \alpha \leq 1$. Let the function $m : [s, x] \to \mathbb{R}$ be continuous and the functions $k_0, k_1 : [0, 1] \times \mathbb{R} \to [0, \infty)$ be continuous and satisfy (2) with m/k_0 and k_1/k_0 Riemann integrable on $[s, x]$. Assume the functions f and g are differentiable as needed. Then
(i) $D^\alpha[af + bg] = aD^\alpha[f] + bD^\alpha[g],\ for\ all\ a, b \in \mathbb{R}$.
(ii) $D^\alpha c = ck_1(\alpha, x),\ for\ all\ constants\ c \in \mathbb{R},\ x \in \mathbb{R}$.
(iii) $D^\alpha[fg] = fD^\alpha[g] + gD^\alpha[f] - fgk_1(\alpha, x),\ x \in \mathbb{R}$.
(iv) $D^\alpha \left[\frac{f}{g}\right] = \frac{gD^\alpha[f]-fD^\alpha[g]}{g^2} + \frac{f}{g}k_1(\alpha, x),\ x \in \mathbb{R}\ and\ g \neq 0$.
(v) For $\alpha \in (0, 1]$ and fixed $s \in \mathbb{R}$, the exponential function satisfies

$$D^\alpha_x[e_m(x, s)] = m(x)e_m(x, s) \tag{6}$$

(vi) For $\alpha \in (0, 1]$ and for the exponential function e_0 given in (5), we have

$$D^\alpha \left[\int_a^x \frac{f(s)e_0(x, s)}{k_0(\alpha, s)} ds\right] = f(x). \tag{7}$$

Proof 1 Using the formula of the modified conformable operator presented in (2.3) we get
(i) For all $a, b \in \mathbb{R}$ we have

$$\begin{aligned}
D^\alpha[af + bg] &= k_1(\alpha, x)(af + bg) + k_0(\alpha, x)\frac{d}{dx}(af + bg) \\
&= a\left(k_1(\alpha, x)f(x) + k_0(\alpha, x)\frac{d}{dx}f(x)\right) + b\left(k_1(\alpha, x)g(x) + k_0(\alpha, x)\frac{d}{dx}g(x)\right) \\
&= aD^\alpha[f] + bD^\alpha[g]
\end{aligned}$$

(ii) For all constants $c \in \mathbb{R},\ x \in \mathbb{R}$ we obtain

$$D^\alpha c = k_1(\alpha, x)c + k_0(\alpha, x)\frac{d}{dx}c = ck_1(\alpha, x).$$

(iii) For all $x \in \mathbb{R}$ we find

$$\begin{aligned}
D^\alpha[fg] &= k_1(\alpha, x)(fg) + k_0(\alpha, x)\frac{d}{dx}(fg) \\
&= k_1(\alpha, x)(fg) + k_0(\alpha, x)\left(g\frac{d}{dx}f + f\frac{d}{dx}g\right) \\
&= f\left(k_1(\alpha, x)g + k_0(\alpha, x)\frac{d}{dx}g\right) + g\left(k_1(\alpha, x)f + k_0(\alpha, x)\frac{d}{dx}g\right) - k_1(\alpha, x)(fg) \\
&= fD^\alpha[g] + gD^\alpha[f] - fgk_1(\alpha, x).
\end{aligned}$$

(iv) For all $x \in \mathbb{R}$ *and* $g \neq 0$ we have

$$\begin{aligned} D^\alpha\left[\frac{f}{g}\right] &= k_1(\alpha,x)\left[\frac{f}{g}\right]+k_0(\alpha,x)\frac{d}{dx}\left[\frac{f}{g}\right] \\ &= k_1(\alpha,x)\left[\frac{f}{g}\right]+\frac{k_0(\alpha,x)\frac{d}{dx}[f]-k_0(\alpha,x)\frac{d}{dx}[g]+gk_1(\alpha,x)f-gk_1(\alpha,x)f}{g^2} \\ &= k_1(\alpha,x)\left[\frac{f}{g}\right]+\frac{g\left(k_0(\alpha,x)\frac{d}{dx}[f]+k_1(\alpha,x)f\right)-f\left(k_0(\alpha,x)\frac{d}{dx}[g]+k_1(\alpha,x)g\right)}{g^2} \\ &= \frac{gD^\alpha[f]-fD^\alpha[g]}{g^2}+\frac{f}{g}k_1(\alpha,x) \end{aligned}$$

(v) For $\alpha \in (0,1]$ and fixed $s \in \mathbb{R}$, the exponential function satisfies

$$\begin{aligned} D_x^\alpha[e_m(x,s)] &= k_1(\alpha,x)\left[e_m(x,s)\right]+k_0(\alpha,x)\frac{\partial}{\partial x}\left[e_m(x,s)\right] \\ &= k_1(\alpha,x)\left[e_m(x,s)\right]+k_0(\alpha,x)\frac{\partial}{\partial x}\left[e^{\int_s^x \frac{m(\lambda)-k_1(\alpha,\lambda)}{k_0(\alpha,\lambda)}d\lambda}\right]e_m(x,s) \\ &= k_1(\alpha,x)\left[e_m(x,s)+k_0(\alpha,x)\frac{m(x)-k_1(\alpha,x)}{k_0(\alpha,x)}\right]e_m(x,s) \\ &= m(x)e_m(x,s). \end{aligned}$$

(vi) For $\alpha \in (0,1]$ and for the exponential function e_0 given in (5), we have

$$\begin{aligned} D^\alpha\left[\int_a^x \frac{f(s)e_0(x,s)}{k_0(\alpha,s)}ds\right] &= k_1(\alpha,x)\left[\int_a^x \frac{f(s)e_0(x,s)}{k_0(\alpha,s)}ds\right]+k_0(\alpha,x)\frac{d}{dx}\left[\int_a^x \frac{f(s)e_0(x,s)}{k_0(\alpha,s)}ds\right] \\ &= k_1(\alpha,x)\left[\int_a^x \frac{f(s)e_0(x,s)}{k_0(\alpha,s)}ds\right] \\ &+ k_0(\alpha,x)\left(\int_a^x \frac{\partial}{\partial x}\left[\frac{f(s)e_0(x,s)}{k_0(\alpha,s)}\right]ds+\frac{f(x)e_0(x,x)}{k_0(\alpha,x)}\right) \\ &= k_1(\alpha,x)\left[\int_a^x \frac{f(s)e_0(x,s)}{k_0(\alpha,s)}ds\right] \\ &+ k_0(\alpha,x)\int_a^x -\frac{k_1(\alpha,x)}{k_0(\alpha,x)}\left[\frac{f(s)e_0(x,s)}{k_0(\alpha,s)}\right]ds+f(x) \\ &= k_1(\alpha,x)\left[\int_a^x \frac{f(s)e_0(x,s)}{k_0(\alpha,s)}ds\right]-k_1(\alpha,x)\left[\int_a^x \frac{f(s)e_0(x,s)}{k_0(\alpha,s)}ds\right]+f(x) \\ &= f(x). \end{aligned}$$

Definition 2.7 (Modified Conformable Integral). [55]
Let $0 < \alpha \leq 1$ and $x_0 \in \mathbb{R}$. In light of (5) and Lemma 2.6 (v) and (vi), define the antiderivative via

$$\int D^\alpha f(x)d_\alpha x = f(x)+ce_0(x,x_0), \quad c \in \mathbb{R}.$$

In the same way define the integral of f over the closed interval $[a,b]$ as follows :

$$\int_a^b f(s)e_0(x,s)d_\alpha s = \int_a^b \frac{f(s)e_0(x,s)}{k_0(\alpha,s)}ds, \quad d_\alpha s = \frac{ds}{k_0(\alpha,s)}. \tag{8}$$

Therefore, we can write:

$$e_0(x,s) = e^{\int_x^s \frac{k_1(\alpha,\lambda)}{k_0(\alpha,\lambda)}d\lambda} = e^{\int_x^s k_1(\alpha,\lambda)d_\alpha\lambda}.$$

Lemma 2.8 (Basic Integrals). [55]
Let the conformable differential operator D^α be given as in (3) and the integral be given as (8) with $0 < \alpha \leq 1$. Let the functions $k_0, k_1 : [0,1] \times \mathbb{R} \to [0, \infty)$ be continuous and satisfy (2) and let f and g be differentiable as needed. Then
(i) The derivative of the definite integral of f is given by

$$D^\alpha \left[\int_a^x f(s) e_0(x,s) d_\alpha s \right] = f(x).$$

(ii) The definite integral of the derivative of f is given by

$$\int_a^x D^\alpha \left[f(s) e_0(x,s) d_\alpha s \right] = f(s) e_0(x,s) \mid_{s=a}^{x} = f(x) - f(a) e_0(x,a).$$

(iii) An integration by parts formula is given as follow

$$\int_a^b f(x) D^\alpha [g(x)] e_0(b,x) d_\alpha x = f(x) g(x) e_0(b,x) \mid_{x=a}^{b} - \int_a^b g(x) (D^\alpha [f(x)] - k_1(\alpha, x) f(x)) e_0(b,x) d_\alpha x.$$

(iv) A version of the Leibniz rule for the differentiation of an integral is given by

$$D^\alpha \left[\int_a^x f(x,s) e_0(x,s) d_\alpha s \right] = \int_a^x \left(D_x^\alpha [f(x,s)] - k_1(\alpha, x) f(x,s) \right) e_0(x,s) d_\alpha s + f(x,x).$$

If $e_0(x,s)$ is absent then by (4) we have

$$D^\alpha \left[\int_a^x f(x,s) d_\alpha s \right] = \int_a^x D_x^\alpha f(x,s) d_\alpha s + f(x,x).$$

Proof 2 It is simple to prove this lemma using the formula of the modified conformable operator presented in (2.3); Lemma 2.6 and integration by part of the usual derivative. More details about this proof can be founded in [55].

In this definition, we will introduce functions that will serve the role that polynomials do in Taylor series expansions for the regular derivative ($\alpha = 1$) which we need in the proof of Theorem 3.1 case 1.

Definition 2.9 Let the functions $\kappa_0, \kappa_1 : [0,1] \times \mathbb{R} \to [0, \infty)$ be continuous such that the conditions in (2.2) are hold. When $\alpha = 1$ and $n \in \mathbb{N}_0$, the polynomials are given by $b_n(x,s) = \frac{1}{n!}(x-s)^n$. To generalize this to the present context, define the functions $b_n : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}, n \in \mathbb{N}_0$ via

$$b_0(x,s) = 1, \quad \forall x, s \in \mathbb{R}$$

and

$$b_n(x,s) = \int_s^x b_{n-1}(\lambda, s) d_\alpha \lambda, \quad n \in \mathbb{N}, \quad \forall x, s \in \mathbb{R}.$$

3. Second Order Linear Modified Conformable Differential Equations (MCDEs)

In this part we will consider the following second order linear homogeneous modified conformable differential equation with constant coefficients

$$a D^\alpha D^\alpha u(x) + b D^\alpha u(x) + c u(x) = 0, \quad x \in [x_0, \infty), \;\; x_0 > 0,$$

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Where a, b, c are real constants. In addition, we will also analyze the related Cauchy - Euler type modified conformable equation

$$axD^{\alpha}[xD^{\alpha}u(x)] + bxD^{\alpha}u(x) + cu(x) = 0, \quad x \in [x_0, \infty), \ \ x_0 > 0,$$

Theorem 3.1 (Constant Coefficients MCDEs)
Let the functions $k_0, k_1 : [0, 1] \times \mathbb{R} \to [0, \infty)$ be continuous and satisfy (2), and let D^{α} be as given in (3). Let $a, b, c \in \mathbb{R}$ be constants and $\alpha \in (0, 1]$. Then the constant coefficients homogeneous modified conformable differential equation

$$aD^{\alpha}D^{\alpha}u(x) + bD^{\alpha}u(x) + cu(x) = 0, \quad x \in [x_0, \infty), \ \ x_0 > 0, \tag{9}$$

has the associated characteristic equation

$$a\lambda^2 + b\lambda + c + 0, \tag{10}$$

and the general solution to (9) is given by one of the following cases:
Case 1: If λ_1, λ_2 are real distinct roots of (10), then

$$u(x) = c_1 e_{\lambda_1}(x, x_0) + c_2 e_{\lambda_2}(x, x_0),$$

Case 2: If λ is a repeated root of (10), then

$$u(x) = c_1 e_{\lambda}(x, x_0) + c_2 e_{\lambda}(x, x_0) \int_{x_0}^{x} 1 d_{\alpha}s,$$

Case 3: If $\lambda = \zeta \pm i\beta$ is a complex root of (10), then

$$u(x) = c_1 e_{\zeta}(x, x_0) \cos\left(\int_{x_0}^{x} \beta d_{\alpha}s\right) + c_2 e_{\zeta}(x, x_0) \sin\left(\int_{x_0}^{x} \beta d_{\alpha}s\right),$$

Proof 3 Let us try the solution

$$u(x) = e_{\lambda}(x, x_0) = e^{\int_{x_0}^{x} \frac{\lambda - k_1(\alpha,t)}{k_0(\alpha,t)} dt}.$$

Substitute $u(x)$ in (9), this leads by (6) to the characteristic equation (10).Thus there are three cases.
Case 1: If $\lambda_1 \neq \lambda_2$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ are the roots of (10), then

$$u(x) = c_1 e_{\lambda_1}(x, x_0) + c_2 e_{\lambda_2}(x, x_0),$$

then

$$D^{\alpha}u(x) = c_1 \lambda_1 e_{\lambda_1}(x, x_0) + c_2 \lambda_2 e_{\lambda_2}(x, x_0),$$

and

$$D^{\alpha}D^{\alpha}u(x) = c_1 \lambda_1^2 e_{\lambda_1}(x, x_0) + c_2 \lambda_2^2 e_{\lambda_2}(x, x_0),$$

Now substitute $u_1(x)$ in (9), we get

$$aD^{\alpha}D^{\alpha}u(x) + bD^{\alpha}u(x) + cu(x) = 0.$$

Since λ_1 and λ_2 are roots of the characteristic equation, we obtain

$$\begin{aligned} c_1 e_{\lambda_1}(x, x_0)(a\lambda_1^2 + b\lambda_1 + c) \ \ &+ \ \ c_2 e_{\lambda_2}(x, x_0)(a\lambda_2^2 + b\lambda_2 + c) \\ &= \ \ c_1 e_{\lambda_1}(x, x_0)(0) + c_2 e_{\lambda_2}(x, x_0)(0) \\ &= \ \ 0. \end{aligned}$$

Case 2: If $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$ is a repeated root of (10), then the first solution is given by

$$u_1(x) = e^{\int_{x_0}^{x} \frac{\lambda - k_1(\alpha,t)}{k_0(\alpha,t)} dt},$$

When $\alpha = 1$ (the classical case) we know that $u_2(x) = x u_1(x)$ is a second linearly independent solution, so we try

$$u_2(x) = u_1(x) b_1(x, x_0) = u_1(x) \int_{x_0}^{x} 1 d_\alpha s = e_\lambda(x, x_0) \int_{x_0}^{x} 1 d_\alpha s.$$

Using Lemma 2.6 (iii) and Lemma 2.8 (iv), we have

$$\begin{aligned} D^\alpha u_2(x) &= D^\alpha \left(e_\lambda(x, x_0) \int_{x_0}^{x} 1 d_\alpha s \right) \\ &= e_\lambda(x, x_0) \left(1 + k_1(\alpha, x) \int_{x_0}^{x} 1 d_\alpha s \right) + \lambda e_\lambda(x, x_0) \int_{x_0}^{x} 1 d_\alpha s \\ &- k_1(\alpha, x) e_\lambda(x, x_0) \int_{x_0}^{x} 1 d_\alpha s \\ &= e_\lambda(x, x_0) \left(1 + \lambda \int_{x_0}^{x} 1 d_\alpha s \right). \end{aligned}$$

Now

$$\begin{aligned} D^\alpha D^\alpha u_2(x) &= D^\alpha \left[e_\lambda(x, x_0) \left(1 + \lambda \int_{x_0}^{x} 1 d_\alpha s \right) \right] \\ &= \lambda e_\lambda(x, x_0) + \lambda \left(e_\lambda(x, x_0) \left(1 + \lambda \int_{x_0}^{x} 1 d_\alpha s \right) \right) \\ &= 2\lambda e_\lambda(x, x_0) + \lambda^2 e_\lambda(x, x_0) \int_{x_0}^{x} 1 d_\alpha s. \end{aligned}$$

With $\lambda = \frac{-b}{2a}$ (since λ is repeated root) and substitute $u_2(x)$ in (9), we get that

$$\begin{aligned} & a \left(2\lambda e_\lambda(x, x_0) + \lambda^2 e_\lambda(x, x_0) \int_{x_0}^{x} 1 d_\alpha s \right) + b e_\lambda(x, x_0) \left(1 + \lambda \int_{x_0}^{x} 1 d_\alpha s \right) + c e_\lambda(x, x_0) \int_{x_0}^{x} 1 d_\alpha s \\ &= e_\lambda(x, x_0) \left(2a\lambda + a\lambda^2 \int_{x_0}^{x} 1 d_\alpha s + b + b\lambda \int_{x_0}^{x} 1 d_\alpha s + c \int_{x_0}^{x} 1 d_\alpha s \right) \\ &= e_\lambda(x, x_0) \left(2a \left(\frac{-b}{2a} \right) + b + a\lambda^2 \int_{x_0}^{x} 1 d_\alpha s + b\lambda \int_{x_0}^{x} 1 d_\alpha s + c \int_{x_0}^{x} 1 d_\alpha s \right) \\ &= e_\lambda(x, x_0) \int_{x_0}^{x} 1 d_\alpha s (a\lambda^2 + b\lambda + c) = 0. \end{aligned}$$

Because λ is a root for the characteristic equation (10). So, $u_2(x)$ is a solution for (9)
Case 3: The roots of (10) complex roots (say $\lambda = \zeta \pm i\beta$). By Euler's formula, we observe that $u(x)$ and the result can found in the classical form as follows:

$$\begin{aligned} u(x) &= e_\zeta(x, x_0) \cos \left(\int_{x_0}^{x} \beta d_\alpha s \right) + i e_\zeta(x, x_0) \sin \left(\int_{x_0}^{x} \beta d_\alpha s \right) \\ &= e_\zeta(x, x_0) \left(\cos \left(\int_{x_0}^{x} \beta d_\alpha s \right) + i \sin \left(\int_{x_0}^{x} \beta d_\alpha s \right) \right) \\ &= e_\zeta(x, x_0) e^{i \int_{x_0}^{x} \beta d_\alpha s} = e^{\int_{x_0}^{x} \zeta d_\alpha s} e^{i \int_{x_0}^{x} \beta d_\alpha s} = e^{\int_{x_0}^{x} \zeta + i\beta d_\alpha s} \\ &= e_{\zeta + i\beta}(x, x_0). \end{aligned}$$

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$$D^{\alpha}u(x) = D^{\alpha}\left[e_{\zeta+i\beta}(x,x_0)\right] = (\zeta+i\beta)e_{\zeta+i\beta}(x,x_0),$$

and

$$\begin{aligned} D^{\alpha}D^{\alpha}u(x) &= D^{\alpha}\left[(\zeta+i\beta)e_{\zeta+i\beta}(x,x_0)\right] \\ &= (\zeta+i\beta)^2 e_{\zeta+i\beta}(x,x_0). \end{aligned}$$

Now substitute $u(x)$ in (9), we get that

$$\begin{aligned} D^{\alpha}D^{\alpha}u(x) &= a(\zeta+i\beta)^2 e_{\zeta+i\beta}(x,x_0) + b(\zeta+i\beta)e_{\zeta+i\beta}(x,x_0) + ce_{\zeta+i\beta}(x,x_0) \\ &= e_{\zeta+i\beta}(x,x_0)(a\lambda^2+b\lambda+c) \\ &= e_{\zeta+i\beta}(x,x_0)(a\lambda^2+b\lambda+c) = 0. \end{aligned}$$

Since λ is a root for the characteristic equation (10).
So the real and imaginary parts of this expression are linearly independent solutions of (9).

The next theorem provided the general solution of the second order Cauchy – Euler modified conformable differential equation.

Theorem 3.2 (Cauchy – Euler MCDEs).
Let the functions $k_0, k_1 : [0,1]\times\mathbb{R} \to [0,\infty)$ be continuous and satisfy (2), and let D^{α} be as given in (3). Let $a,b,c \in \mathbb{R}$ be constants and $\alpha \in (0,1]$. Then the homogeneous Cauchy – Euler type modified conformable differential equation (MCDE)

$$axD^{\alpha}\left[xD^{\alpha}u(x)\right] + bD^{\alpha}u(x) + cu(x) = 0, \quad x \in [x_0,\infty),\ \ x_0 > 0, \tag{11}$$

has the associated characteristic equation (10) and the general solution to (11) is given by one of the following cases for the constants $c_1, c_2 \in \mathbb{R}$
Case 1: If $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 \neq \lambda_2$ are distinct roots of (10), then the general solution is given by

$$u(x) = c_1 e_{\lambda_1/x}(x,x_0) + c_2 e_{\lambda_2/x}(x,x_0).$$

Case 2: If $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\lambda_1 = \lambda_2 = \lambda = \frac{-b}{2a}$ are repeated roots of (10), then the general solution is given by

$$u(x) = c_1 e_{\lambda/x}(x,x_0) + c_2 e_{\lambda/x}(x,x_0)\int_{x_0}^{x} s^{-1} d_{\alpha}s.$$

Case 3: If $\lambda = \zeta \pm i\beta$ is a complex root of (10), then the general solution is given by

$$u(x) = c_1 e_{\zeta/x}(x,x_0)\cos\left(\beta\int_{x_0}^{x} s^{-1}d_{\alpha}s\right) + c_2 e_{\zeta/x}(x,x_0)\sin\left(\beta\int_{x_0}^{x} s^{-1}d_{\alpha}s\right).$$

Proof 4 Case 1: Let‘s begin with $u_1(x)$, and similarly for $u_2(x)$.

$$\begin{aligned} D^{\alpha}u_1(x) &= D^{\alpha}\left(e_{\lambda_1/x}(x,x_0)\right) \\ &= \frac{\lambda_1}{x}e_{\lambda_1/x}(x,x_0). \end{aligned}$$

Consequently, we have

$$\begin{aligned} D^{2\alpha}u_1(x) &= D^{\alpha}D^{\alpha}u_1(x) \\ &= D^{\alpha}\left(\frac{\lambda_1}{x}e_{\lambda_1/x}(x,x_0)\right) \\ &= \frac{\lambda_1^2}{x^2}e_{\lambda_1/x}(x,x_0) - \frac{\lambda_1}{x^2}k_0(\alpha,x)e_{\lambda_1/x}(x,x_0). \end{aligned}$$

Now substitute $u_1(x)$ in (11), we see that

$$\begin{aligned}
&ax\left[\frac{\lambda_1^2}{x^2}e_{\lambda_1/x}(x,x_0)-\frac{\lambda_1}{x^2}k_0(\alpha,x)e_{\lambda_1/x}(x,x_0)+D^\alpha u_1(x)(k_0(\alpha,x)+k_1(\alpha,x)x)\right.\\
&\quad\left.-k_1(\alpha,x)xD^\alpha u_1(x)\right]+bx\left(\frac{\lambda_1}{x}e_{\lambda_1/x}(x,x_0)\right)+ce_{\lambda_1/x}(x,x_0)\\
&=a\lambda_1^2e_{\lambda_1/x}(x,x_0)-a\lambda_1k_0(\alpha,x)e_{\lambda_1/x}(x,x_0)+a\lambda_1k_0(\alpha,x)e_{\lambda_1/x}(x,x_0)\\
&\quad+ax\lambda_1k_1(\alpha,x)e_{\lambda_1/x}(x,x_0)-ax\lambda_1k_1(\alpha,x)e_{\lambda_1/x}(x,x_0)\\
&\quad+b\lambda_1xe_{\lambda_1/x}(x,x_0)+ce_{\lambda_1/x}(x,x_0)\\
&=a\lambda_1^2e_{\lambda_1/x}(x,x_0)+b\lambda_1xe_{\lambda_1/x}(x,x_0)+ce_{\lambda_1/x}(x,x_0)\\
&=e_{\lambda_1/x}(x,x_0)(a\lambda^2+b\lambda+c)\\
&=0.
\end{aligned}$$

Since, λ_1 is a root for the characteristic equation (10).
Case 2: If $\lambda=\frac{-b}{2a}$ is repeated root and

$$u_2(x)=u_1(x)\int_{x_0}^{x}s^{-1}d_\alpha s=e_{\lambda/x}(x,x_0)\int_{x_0}^{x}s^{-1}d_\alpha s,$$

then

$$\begin{aligned}
D^{2\alpha}u_2(x) &= D^\alpha\left(e_{\lambda/x}(x,x_0)\int_{x_0}^{x}s^{-1}d_\alpha s\right)\\
&= e_{\lambda/x}(x,x_0)\left(\frac{1}{x}k_1(\alpha,x)\int_{x_0}^{x}s^{-1}d_\alpha s\right)\\
&+ D^\alpha e_{\lambda/x}(x,x_0)\int_{x_0}^{x}s^{-1}d_\alpha s-k_1(\alpha,x)e_{\lambda/x}(x,x_0)\int_{x_0}^{x}s^{-1}d_\alpha s\\
&= \frac{u_1(x)}{x}+k_1(\alpha,x)u_1(x)\int_{x_0}^{x}s^{-1}d_\alpha s+D^\alpha u_1(x)\int_{x_0}^{x}s^{-1}d_\alpha s\\
&- k_1(\alpha,x)e_{\lambda/x}(x,x_0)\int_{x_0}^{x}s^{-1}d_\alpha s\\
&= \frac{u_1(x)}{x}+D^\alpha u_1(x)\int_{x_0}^{x}s^{-1}d_\alpha s.
\end{aligned}$$

$$\begin{aligned}
D^{2\alpha}u_2(x) &= D^\alpha D^\alpha u_2(x)\\
&= D^\alpha\left[c+D^\alpha u_1(x)\int_{x_0}^{x}s^{-1}d_\alpha s\right]\\
&= \frac{D^\alpha u_1(x)}{x}+u_1(x)\left(\frac{k_1(\alpha,x)}{x}-\frac{k_0(\alpha,x)}{x^2}\right)-\frac{k_1(\alpha,x)u_1(x)}{x}\\
&+ D^\alpha u_1(x)\left(\frac{1}{x}k_1(\alpha,x)\int_{x_0}^{x}s^{-1}d_\alpha s\right)\\
&+ D^{2\alpha}u_1(x)\int_{x_0}^{x}s^{-1}d_\alpha s-k_1(\alpha,x)D^\alpha u_1(x)\int_{x_0}^{x}s^{-1}d_\alpha s\\
&= \frac{2D^\alpha u_1(x)}{x}-\frac{k_0(\alpha,x)u_1(x)}{x^2}+D^{2\alpha}u_1(x)\int_{x_0}^{x}s^{-1}d_\alpha s
\end{aligned}$$

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Now substitute $u_2(x)$ in (11) we get that

$$\begin{aligned}
& axD^{\alpha}[xD^{\alpha}u_2(x)] + bD^{\alpha}u_1(x)\int_{x_0}^{x} s^{-1}d_{\alpha}s + cu_1(x)\int_{x_0}^{x} s^{-1}d_{\alpha}s \\
= \;& ax[x(D^{2\alpha}u_2(x)) + xk_1(\alpha,x)D^{\alpha}u_2(x) \\
+ \;& D^{\alpha}u_2(x)k_0(\alpha,x) - k_1(\alpha,x)D^{\alpha}u_2(x)] + bD^{\alpha}u_2(x) + cu_2(x) \\
= \;& ax[x\left(\frac{2D^{\alpha}u_1(x)}{x} - \frac{k_0(\alpha,x)u_1(x)}{x^2} + D^{2\alpha}u_1(x)\int_{x_0}^{x} s^{-1}d_{\alpha}s\right) \\
+ \;& k_0(\alpha,x)\left(\frac{u_1(x)}{x} + D^{\alpha}u_1(x)\int_{x_0}^{x} s^{-1}d_{\alpha}s\right)] \\
+ \;& bx\left(\frac{u_1(x)}{x} + D^{\alpha}u_1(x)\int_{x_0}^{x} s^{-1}d_{\alpha}s\right) + cu_1(x)\int_{x_0}^{x} s^{-1}d_{\alpha}s \\
= \;& 2axD^{\alpha}u_1(x) + ax^2D^{2\alpha}u_1(x)\int_{x_0}^{x} s^{-1}d_{\alpha}s + axk_0(\alpha,x)D^{\alpha}u_1(x)\int_{x_0}^{x} s^{-1}d_{\alpha}s \\
+ \;& bu_1(x) + bxD^{\alpha}u_1(x)\int_{x_0}^{x} s^{-1}d_{\alpha}s + cu_1(x)\int_{x_0}^{x} s^{-1}d_{\alpha}s \\
= \;& 2ax\left(\frac{-b}{2a}\right)u_1(x) + ax[xD^{2\alpha}u_1(x)\int_{x_0}^{x} s^{-1}d_{\alpha}s \\
+ \;& k_0(\alpha,x)D^{\alpha}u_1(x)\int_{x_0}^{x} s^{-1}d_{\alpha}s] + bu_1(x) + bxD^{\alpha}u_1(x)\int_{x_0}^{x} s^{-1}d_{\alpha}s + cu_1(x)\int_{x_0}^{x} s^{-1}d_{\alpha}s \\
+ \;& bxD^{\alpha}u_1(x)\int_{x_0}^{x} s^{-1}d_{\alpha}s + cu_1(x)\int_{x_0}^{x} s^{-1}d_{\alpha}s \\
= \;& \left[ax\left(xD^{2\alpha}u_1(x) + k_0(\alpha,x)D^{\alpha}u_1(x)\right) + bxD^{\alpha}u_1(x) + cu_1(x)\right]\int_{x_0}^{x} s^{-1}d_{\alpha}s \\
= \;& [axD^{\alpha}\left(xD^{\alpha}u_1(x)\right) + bxD^{\alpha}u_1(x) + cu_1(x)]\int_{x_0}^{x} s^{-1}d_{\alpha}s = 0.
\end{aligned}$$

Since $u_1(x)$ is a solution for (10), we conclude that $u_2(x)$ is a solution for (10).
Case 3: If $\lambda = \zeta + i\beta$ is a complex root of (10), then the complex for (11) solution is given by

$$\begin{aligned}
u(x) &= e_{(\zeta+i\beta)/x}(x, x_0) \\
&= e_{\zeta/x}(x, x_0)e^{i\beta/x} \\
&= e_{\zeta/x}(x, x_0)\left[\cos\left(\beta\int_{x_0}^{x} s^{-1}d_{\alpha}s\right) + i\sin\left(\beta\int_{x_0}^{x} s^{-1}d_{\alpha}s\right)\right].
\end{aligned}$$

And again both real and imaginary parts of the above expression are linearly independent solutions of (11).

4. Applications

As an application we are going to illustrate the method on some numerical examples. It will be raised in the form of problems that will handled.
Problem 1:
Consider the following linear second order constant coefficients modified conformable differential equation:

$$D^{\alpha}D^{\alpha}u(x) - 2D^{\alpha}u(x) + u(x) = 0, \quad x \in [1, \infty), \tag{12}$$

With initial conditions

$$u(1) = 4, \quad D^{\alpha}u(1) = 5. \tag{13}$$

Assume that $k_0(\alpha, x), k_1(\alpha, x)$ satisfy (2). If $k_1(\alpha, x)$ is differentiable on $[1, \infty)$ and $\alpha \in [0, 1)$ (condition of Theorem 3.1) then,
We observe that characteristic equation for the modified conformable differential equation (4.1) is given by

$$\lambda^2 - 2\lambda + 1,$$

and the roots of it are $\lambda = 1$ repeated, so by Theorem 3.1 the general solution of the equation (4.1) is as follow:

$$u(x) = c_1 e_1(x, 1) + c_2 e_1(x, 1) \int_1^x 1 d_\alpha s,$$

Where c_1 and c_2 are constants.
The initial conditions (4.2) and the use of Lemma 2.6 (iii) and Lemma 2.8 (iv), implies that

$$\begin{aligned} u(1) &= \left(c_1 e_1(x, 1) + c_2 e_1(x, 1) \int_1^x 1 d_\alpha s \right) |_{x=1} \\ &= c_1 e_1(1, 1) + c_2 e_1(1, 1) \int_1^1 1 d_\alpha s = c_1 e_1(1, 1) + 0 \\ &= c_1 e_1(1, 1) = c_1 e^{\int_1^1 \frac{m(\lambda) - k_1(\alpha, \lambda)}{k_0(\alpha, \lambda)} d\lambda} = c_1 e^0 = c_1 \\ &= 4, \end{aligned}$$

Also,

$$\begin{aligned} D^\alpha u(1) &= D^\alpha \left(c_1 e_1(x, 1) + c_2 e_1(x, 1) \int_1^x 1 d_\alpha s \right) |_{x=1} \\ &= D^\alpha (c_1 e_1(x, 1)) |_{x=1} + D^\alpha \left(c_2 e_1(x, 1) \int_1^x 1 d_\alpha s \right) |_{x=1} \quad \textit{by Lemma } 2.6\ (i) \\ &= D^\alpha (4 e_1(x, 1)) |_{x=1} + c_2 e_1(x, 1) \left(1 + \int_1^x 1 d_\alpha s \right) |_{x=1} \quad \textit{by Lemma } 2.6\ (iii) \\ &\qquad \textit{and Lemma } 2.8\ (iv) \\ &= D^\alpha (4 e_1(x, 1)) |_{x=1} + c_2 e_1(1, 1) \left(1 + \int_1^1 1 d_\alpha s \right) \\ &= D^\alpha (4 e_1(x, 1)) |_{x=1} + c_2 e_1(1, 1) \\ &= D^\alpha (4 e_1(x, 1)) |_{x=1} + c_2 \quad \textit{by Lemma } 2.6\ (v) \\ &= 4 e_1(1, 1) + c_2 = 4 + c_2 = 5. \end{aligned}$$

Hence $c_2 = 1$. Therefore, the exact solution for the modified conformable differential equation (4.1) is given by

$$u(x) = 4 e_1(x, 1) + e_1(x, 1) \int_1^x 1 d_\alpha s.$$

Problem 2:
We present the second constant coefficients modified conformable differential equations as can be seen bellow:

$$D^\alpha D^\alpha u(x) - 2 D^\alpha u(x) + 2 u(x) = 0, \quad x \in [2, \infty), \tag{14}$$

With initial conditions

$$u(2) = 1, \quad D^\alpha u(2) = 2. \tag{15}$$

Assume that $k_0(\alpha, x), k_1(\alpha, x)$ satisfy (2). If $k_1(\alpha, x)$ is differentiable on $[1, \infty)$ and $\alpha \in [0, 1)$ (condition of Theorem 3.1). So, it is clear that the associated characteristic equation for the modified conformable differential equation (4.3) is given by

$$\lambda^2 - 2\lambda + 2 = 0,$$

and the roots of it are $\lambda = 1 \pm i$ (complex roots). So, based on Theorem 3.1, the general solution is given by

$$u(x) = c_1 e_1(x, 2) \cos\left(\int_2^x 1 d_\alpha s\right) + c_2 e_1(x, 2) \sin\left(\int_2^x 1 d_\alpha s\right),$$

for some constants c_1, c_2.
The initial conditions (4.4) and the use of Lemma 2.6 (iii) and Lemma 2.8 (iv), implies that

$$\begin{aligned} u(2) &= \left[c_1 e_1(x, 2) \cos\left(\int_2^x 1 d_\alpha s\right) + c_2 e_1(x, 2) \sin\left(\int_2^x 1 d_\alpha s\right)\right] |_{x=2} \\ &= c_1 e_1(2, 2) \cos\left(\int_2^2 1 d_\alpha s\right) + c_2 e_1(2, 2) \sin\left(\int_2^2 1 d_\alpha s\right) \\ &= c_1 e_1(2, 2) \cos(0) + c_2 e_1(2, 2) \sin(0) \\ &= c_1 e^{\int_2^2 \frac{m(\lambda) - k_1(\alpha, \lambda)}{k_0(\alpha, \lambda)} d\lambda} \\ &= c_1 = 1. \end{aligned}$$

Also,

$$\begin{aligned} D^\alpha u(2) &= D^\alpha \left[c_1 e_1(x, 2) \cos\left(\int_2^x 1 d_\alpha s\right) + c_2 e_1(x, 2) \sin\left(\int_2^x 1 d_\alpha s\right)\right] |_{x=2} \\ &= c_1 D^\alpha \left[e_1(x, 2) \cos\left(\int_2^x 1 d_\alpha s\right)\right] |_{x=2} + c_2 D^\alpha \left[e_1(x, 2) \sin\left(\int_2^x 1 d_\alpha s\right)\right] |_{x=2} \\ &\qquad \textit{by Lemma } 2.6\ (i) \\ &= c_1 \left[e_1(x, 2) D^\alpha \left(\cos\left(\int_2^x 1 d_\alpha s\right)\right) + \cos\left(\int_2^x 1 d_\alpha s\right) D^\alpha (e_1(x, 2))\right] |_{x=2} \\ &- c_1 \left[e_1(x, 2) \cos\left(\int_2^x 1 d_\alpha s\right) k_1(\alpha, x)\right] |_{x=2} \\ &+ c_2 \left[e_1(x, 2) D^\alpha \left(\sin\left(\int_2^x 1 d_\alpha s\right)\right) + \sin\left(\int_2^x 1 d_\alpha s\right) D^\alpha (e_1(x, 2))\right] |_{x=2} \\ &- c_2 \left[e_1(x, 2) \sin\left(\int_2^x 1 d_\alpha s\right) k_1(\alpha, x)\right] |_{x=2} \\ &\qquad \textit{by Lemma } 2.6\ (i) \textit{ and } (v) \textit{ and Lemma } 2.8\ (iv) \\ &= c_1 [k_1(\alpha, 2) + 1 - k_1(\alpha, 2)] + c_2 [1 + 0 - 0] \\ &= c_1 + c_2 = 2. \end{aligned}$$

Since $c_1 = 1$, we deduce that $c_2 = 1$. Consequently, we conclude that the exact solution of the second order constant coefficients modified conformable differential equation (4.3) is giving as

$$u(x) = e_1(x, 2) \cos\left(\int_2^x 1 d_\alpha s\right) + e_1(x, 2) \sin\left(\int_2^x 1 d_\alpha s\right).$$

Problem 3:
Consider the following second order Cauchy-Euler modified conformable differential equation

$$x D^\alpha [x D^{2\alpha} u(x)] - 4u(x) = 0, \qquad x \in [3, \infty), \tag{16}$$

subject to the following initial conditions

$$u(3) = 2, \quad D^{\alpha}u(3) = 4. \tag{17}$$

Assume that $k_0(\alpha, x), k_1(\alpha, x)$ satisfy (2). If $k_1(\alpha, x)$ is differentiable on $[1, \infty)$ and $\alpha \in [0, 1)$ (condition of Theorem 3.2) then,
Notice that the associated characteristic equation of the previous Cauchy-Euler modified conformable differential equation is given by

$$\lambda^2 - 4 = 0,$$

and the roots of it are

$$\lambda = \pm 2.$$

So, by Theorem 3.2 the general solution of the Cauchy-Euler modified conformable differential equation (4.5) is giving as

$$u(x) = c_1 e_{\frac{2}{x}}(x, 3) + c_2 e_{\frac{-2}{x}}(x, 3),$$

Where c_1 and c_2 are constants to be determined based on the initial conditions.
The initial conditions (4.6) and the use Lemma 2.6 (iii) and Lemma 2.8 (iv), show that

$$\begin{aligned} u(3) &= \left(c_1 e_{\frac{2}{x}}(x,3) + c_2 e_{\frac{-2}{x}}(x,3)\right)|_{x=3} \\ &= c_1 e_{\frac{2}{3}}(3,3) + c_2 e_{\frac{-2}{3}}(3,3) \\ &= c_1 e^{\int_3^3 \frac{m(\lambda)-k_1(\alpha,\lambda)}{k_0(\alpha,\lambda)} d\lambda} + c_2 e^{\int_3^3 \frac{m(\lambda)-k_1(\alpha,\lambda)}{k_0(\alpha,\lambda)} d\lambda} \\ &= c_1 + c_2 = 2. \end{aligned}$$

In other side, using Lemma 2.6 (i) and Lemma 2.8 (iv), we get

$$\begin{aligned} D^{\alpha}u(3) &= D^{\alpha}\left(c_1 e_{\frac{2}{x}}(x,3) + c_2 e_{\frac{-2}{x}}(x,3)\right)|_{x=3} \\ &= \left(c_1 \frac{2}{x} e_{\frac{2}{x}}(x,3) + c_2 \frac{-2}{x} e_{\frac{-2}{x}}(x,3)\right)|_{x=3} \\ &= c_1 \frac{2}{3} e_{\frac{2}{3}}(3,3) + c_2 \frac{-2}{3} e_{\frac{-2}{3}}(3,3) \\ &= \frac{2}{3}c_1 - \frac{2}{3}c_2 = 4. \end{aligned}$$

Therefore we get $\begin{cases} c_1 + c_2 = 2, \\ \frac{2}{3}c_1 - \frac{2}{3}c_2 = 4, \end{cases} \Rightarrow \begin{cases} c_2 = 2 - c_1, \\ \frac{2}{3}c_1 - \frac{2}{3}c_2 = 4, \end{cases}$.
Thus

$$\frac{2}{3}c_1 - \frac{2}{3}(2 - c_1) = 4 \Rightarrow \frac{2c_1 + 2c_1 - 4}{3} = \frac{12}{3} \Rightarrow \frac{4c_1}{3} = \frac{16}{3}$$

$$c_1 = \frac{16}{4} = 4, \quad then \;\; c_2 = -2$$

Therefore, we conclude that the exact solution of the second order Cauchy-Euler modified conformable differential equation (4.5) is presented as

$$u(x) = 4e_{\frac{2}{x}}(x, 3) - 2e_{\frac{-2}{x}}(x, 3).$$

Hence a result as required.

5. Conclusion

In this work, the modified conformable operator D^α has been successfully utilized to obtain exact solutions for second-order differential equations with constant coefficients, including their associated Cauchy-Euler forms. The presented technique has proven effective, as demonstrated by numerical examples, confirming its reliability in solving such equations. Future research may extend this approach to higher-order and nonlinear differential equations, as well as explore broader applications in physics and engineering.

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